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Exact half-BPS type IIB interface solutions II: flux solutions and multi-Janus

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Abstract: Regularity and topology conditions are imposed on the exact Type IIB solutions on $AdS_4 \times S^2 \times S^2 \times \Sigma$ with 16 supersymmetries, which were derived in a companion paper [1]. We construct an infinite class of regular solutions with varying dilaton, and non-zero 3-form fluxes. Our solutions may be viewed as the fully back-reacted geometries of $AdS_5 \times S^5$ (or more generally, Janus) doped with D5 and/or NS5 branes. The solutions are parametrized by the choice of an arbitrary genus g hyper-elliptic Riemann surface Σ with boundary, all of whose branch points are restricted to lie on a line. For genus 0, the Janus solution with 16 supersymmetries and 6 real parameters is recovered; its topology coincides with that of $AdS_5 \times S^5$. The genus $g \geq 1$ solutions are parametrized by a total of $4g+6$ real numbers, $2g-1$ of which are the real moduli of Σ. The solutions have $2g+2$ asymptotic $AdS_5 \times S^5$ regions, g three-spheres with RR 3-form charge, and another g with NSNS 3-form charge. Collapse of consecutive branch points of Σ yields singularities which correspond to D5 and NS5 branes in the probe limit. It is argued that the AdS/CFT dual gauge theory to each of our solutions consists of a $2+1$-dimensional planar interface on which terminate $2g+2$ half-Minkowski 3+1-dimensional space-time $\mathcal{N}=4$ super-Yang-Mills theories. Generally, the $\mathcal{N}=4$ theory in each Minkowski half-space-time may have an independent value of the gauge coupling, and the interface may support various operators, whose interface couplings are further free parameters of the dual gauge theory.

Keywords: AdS-CFT Correspondence, Brane Dynamics in Gauge Theories.

Contents

1. Introduction

In a companion paper [1], the complete solution with 16 supersymmetries was obtained for Type IIB supergravity on $AdS_4 \times S^2 \times S^2 \times \Sigma$ with SO(2,3)$\times$SO(3)$\times$SO(3) isometry.[1] The solutions of [1] were found analytically in terms of two locally harmonic functions h_1 and h_2 on a Riemann surface Σ with boundary. Generally, these solutions have varying dilaton and non-vanishing 3-form RR and NSNS fluxes. The goal of this paper is to present the construction of an infinite subclass of such solutions which have non-singular geometry.

One motivation for this investigation derives from the fact that the AdS/CFT duals to planar interface super-Yang-Mills theories in four dimensions are of this type. In particular, a planar interface theory with 16 supersymmetries was predicted to exist in [3], and its AdS/CFT dual was indeed found to be a regular supersymmetric Janus solution in [1]. (The related problem of AdS/CFT duals to defect super-Yang-Mills theory in the probe

[1] The corresponding BPS equations for this geometry were obtained in [2], but no solutions, other than $AdS_5 \times S^5$, were constructed there.

limit was studied in [4].) Another motivation stems from the similarity of the problem with the construction of half-BPS "bubbling geometries" in [5], and of AdS/CFT duals to half-BPS Wilson loop operators in $\mathcal{N} = 4$ super Yang-Mills in 4-dimensional space-time [6−8]. In particular, the AdS/CFT dual to a half-BPS Wilson loop corresponds to the geometry $AdS_2 \times S^2 \times S^4 \times \Sigma$ (see [6]); its general solution will be obtained in a further companion paper [9]. A final motivation is the construction of fully back-reacted Type IIB supergravity solutions in which the $AdS_5 \times S^5$ geometry is doped with D5 and/or NS5 branes.

In the present paper, we shall derive a set of regularity and topology conditions under which some of the general solutions of [1] are non-singular. We shall restrict attention to solutions for which one S^2, or the other S^2, shrinks to zero size on the boundary $\partial\Sigma$ of Σ. This will partition the boundary $\partial\Sigma$ into segments, each segment corresponding to the vanishing of either one S^2, or the other S^2, but not both. While the segments lie on the boundary $\partial\Sigma$, they correspond to regular interior points of the full 10-dimensional geometry. Two consecutive segments meet at a point on $\partial\Sigma$, which actually corresponds to an asymptotic $AdS_5 \times S^5$ throat region.

A general class of regular solutions allows for an arbitrary even number $2g + 2$ of asymptotic $AdS_5 \times S^5$ regions, connected by a smooth geometry. Specifically, these solutions will be parametrized by a genus $g \geq 0$ hyperelliptic Riemann surface Σ with boundary $\partial\Sigma$, whose $2g+2$ branch points lie on the real line. The harmonic functions h_1 and h_2 will obey alternating Neumann and Dirichlet boundary conditions on $\partial\Sigma$, corresponding to whether one S^2 or the other S^2 shrinks to zero radius. The regularity and topology conditions, discussed in the preceding paragraph, will impose constraints on the zeros and poles of the Abelian differentials ∂h_1 and ∂h_2. In particular, ∂h_1 and ∂h_2 will have a double pole at each branch point, turning the point into an asymptotic $AdS_5 \times S^5$ region. Regularity requires a specific ordering pattern of the real zeros of the Abelian differentials ∂h_1 and ∂h_2 with respect to the branch points. These conditions will be solved in this paper. The resulting solutions will be referred to as "multi-Janus" solutions.

Our half-BPS solutions have some resemblance to the "bubbling AdS" solutions found in [5]. The "coloring" of [5] would correspond here to the alternating Neumann and Dirichlet boundary conditions obeyed by h_1 and h_2 on $\partial\Sigma$. From this perspective, the solutions obtained in this paper might be referred to as "bubbling multi-Janus" solutions.

At genus $g = 0$, the Janus solution with 16 supersymmetries with 6 free parameters, first obtained in [1], is recovered. It indeed exhibits 2 asymptotic $AdS_5 \times S^5$ regions, in agreement with the above counting, and has the topology of $AdS_5 \times S^5$.

At genus $g \geq 1$, the regular solutions are parametrized by the $2g - 1$ real moduli of Σ, and the $2g + 2$ real zeros of the Abelian differentials ∂h_1 and ∂h_2. Each solution has five further real parameters; one for the overall scale of the dilaton; one for the overall scale of the 10-dimensional metric, and 3 others allowing for global $SU(1,1)$ S-duality rotations to the general solution with a non-zero axion field. The total number of free parameters is thus $4g + 6$. The differentials ∂h_1 and ∂h_2 may also have common pairs of complex conjugate zeros, but their positions are fixed in terms of the moduli and the real zeros of ∂h_1 and ∂h_2 by certain elliptic or hyperelliptic period relations, which will be presented

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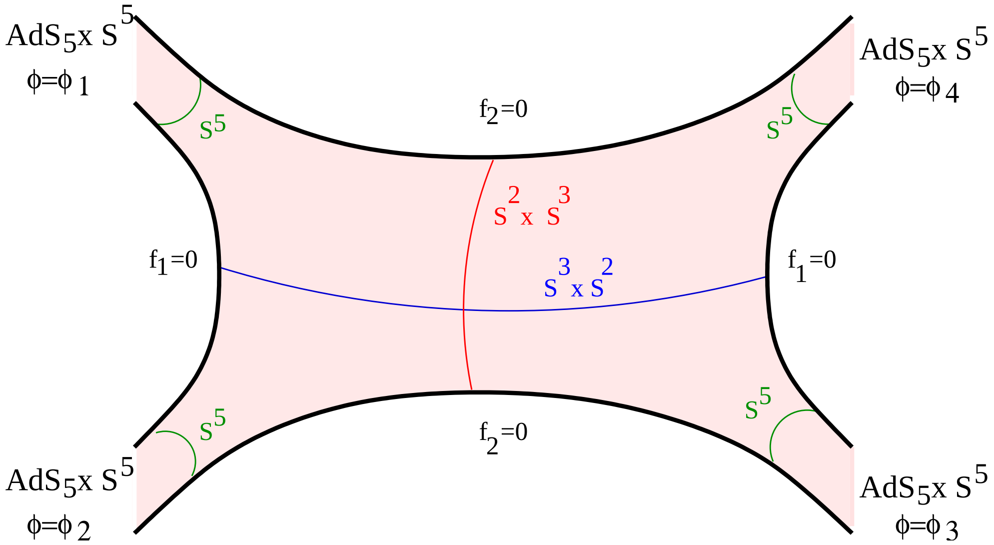

Figure 1: The genus 1 solution has 4 distinct asymptotic $AdS_5 \times S^5$ regions, each with a different constant limit $\phi_1, \phi_2, \phi_3, \phi_4$ for the dilaton field. The radii f_1 and f_2 of the 2-spheres vanish on alternating segments of the boundary. The locations of the homology 3-spheres corresponding to RR and NSNS charges are also indicated.

explicitly.

The genus $g \geq 1$ solutions have $2g+2$ asymptotic $AdS_5 \times S^5$ regions. In each of these asymptotic regions, the dilaton tends to a constant, but the constants for different $AdS_5 \times S^5$ regions will generally be different from one another. The complete boundary of the full 10-dimensional geometry consists of $2g+2$ four-dimensional conformal Minkowski space-times, each of which is obtained as the boundary in each separate $AdS_5 \times S^5$ region. The AdS/CFT correspondence implies that a conformal field theory will live in each of these $2g+2$ four-dimensional Minkowski space-times. On the inside of the 10-dimensional geometry, we find g homology 3-spheres which carry non-vanishing RR 3-form charge (and vanishing NSNS 3-form charge), and another g homology spheres which carry non-vanishing NSNS 3-form charge (and vanishing RR 3-form charge).

The genus $g = 1$ case will be worked out in complete detail in terms of elliptic functions. The global geometry of the solution indeed exhibits 4 asymptotic $AdS_5 \times S^5$ regions, each with a distinct constant limiting value of the dilaton field, as represented in figure 1. Existence of regular solutions will be shown analytically, for parameters valued in some open neighborhood.

For higher genus g, we present an argument for the existence of regular solutions by induction on the genus g. (The argument applies when the free parameters are valued in an open neighborhood containing the parameter space of the genus $g-1$ regular solutions.) We have confirmed the existence of regular genus 2 solutions also by numerical analysis.

There is a sense in which the families of regular solutions for different genera g are connected to one another. We shall show that one may pass from a regular genus g solution to a regular genus $g-1$ solution by allowing a branch cut between two consecutive branch points to shrink to zero size. The regularity and topology conditions require that the collapse of a branch cut on Σ always be accompanied by the convergence to this branch cut of one of the complex zeros, and one real zero of either ∂h_1 or ∂h_2. A first option is to also let a second real zero converge to the collapsing branch cut; the regular solution at genus $g-1$ is thereby recovered. A second option is to leave the remaining real zeros arbitrary. The collapsing branch cut then leaves two simple poles behind, which are argued to correspond to a naked D5 or NS5 brane. The limit where the residue of this pole tends to 0 yields these D5 and NS5 branes in the probe limit. The solution with g collapsed branch cuts is derived explicitly and shown to correspond to a Janus geometry doped with g naked D5/NS5 branes.

The global description of the genus g parameter space of regular solutions will be presented in the form of the definite ordering prescription of the branch points of Σ and the real zeros of the differentials ∂h_1 and ∂h_2, as well as the vanishing of the Abelian integrals of ∂h_1 and ∂h_2 between certain consecutive branch points. The analysis of the global structure and topology of these parameter spaces poses an exciting and challenging mathematical problem, which is comparable to the problem of the studying the moduli space of instantons and magnetic monopoles.

The remainder of this paper is organized as follows. In section 2, we review the Ansatz for the Type IIB supergravity fields corresponding to the geometry $AdS_4 \times S^2 \times S^2 \times \Sigma$ with $SO(2,3) \times SO(3) \times SO(3)$ isometry, as well as the form of the complete exact solution in terms of two locally harmonic functions h_1 and h_2 on Σ, derived in [1]. In section 3, we derive the general regularity and topology conditions on h_1, h_2, and their differentials ∂h_1 and ∂h_2. In section 4, we introduce our general solution by presenting the genus g hyperelliptic Ansatz, and subjecting it to all the regularity and topology conditions of section 3. These conditions are solved in terms of a unique relative ordering of the branch points of Σ and the real zeros of the differentials ∂h_1 and ∂h_2. For $g \geq 1$, the homology 3-spheres are constructed and the corresponding charges of the RR and NSNS 3-forms are evaluated and shown to be generally non-vanishing. In section 5, the genus 1 case is worked out in complete detail. In section 6, the genus 2 case is solved in part analytically, and in part numerically, and a general argument is given for the existence of solutions to all genera. In section 7, the collapse of branch cuts is carried out, and the presence of probe D5/NS5 branes is demonstrated by evaluating the 3-form fluxes on these solutions. In section 8, a dual gauge theory is proposed as a generalization of the interface conformal field theory discussed in [3]. In appendix A we add poles to the differentials ∂h_1 and ∂h_2 that are on the inside of Σ, and show that these do not lead to regular solutions.

2. Review of the Ansatz and the general local solution

In Type IIB supergravity [10, 11], the bosonic fields are the metric ds^2, the axion/dilaton fields, represented by the 1-forms P and Q, the complex 3-form G and the self-dual 5-form $F_{(5)}$. Upon use of the Bianchi identities, the field P, Q may equivalently be represented by the more customary dilaton and axion fields Φ and χ, while the 3-form G and the 5-form

$F_{(5)}$ derive from the complex 2-form $B_{(2)}$ and real 4-form $C_{(4)}$ potentials. The fermionic fields are the gravitino ψ_M and the dilatino λ. The field equations, and BPS equations $\delta\psi_M = \delta\lambda = 0$ for $\psi_M = \lambda = 0$, as well as our conventions for Dirac matrices were presented in detail in [1].

In this section, we shall review the Ansatz for the fields on the $AdS_4 \times S_1^2 \times S_2^2 \times \Sigma$ geometry with $SO(2,3) \times SO(3) \times SO(3)$ isometry, as well as the complete solution with 16 supersymmetries derived in [1].

2.1 The Ansatz

The Ansatz for the metric is

$$ds^2 = f_4^2 ds^2_{AdS_4} + f_1^2 ds^2_{S_1^2} + f_2^2 ds^2_{S_2^2} + ds^2_\Sigma \tag{2.1}$$

Here, $ds^2_{AdS_4}$, $ds^2_{S_1^2}$ and $ds^2_{S_2^2}$ are the maximally symmetric metrics respectively on AdS_4, S_1^2 and S_2^2 with unit radii; ds^2_Σ is a Riemannian metric on Σ, and f_1, f_2, f_4 are real functions on Σ. Since Σ has a metric and an orientation it is a Riemann surface, and we may choose local conformal complex coordinates $w, \bar{w}$, in which $ds^2_\Sigma = 4\rho^2 |dw|^2$, for a real ρ on Σ. It will be useful to recast the frame e^a, $a = 8, 9$ on Σ in terms of local complex coordinates,[2] and

$$\begin{aligned} e^z &= (e^8 + ie^9)/2 = \rho dw \\ e^{\bar{z}} &= (e^8 - ie^9)/2 = \rho d\bar{w} \end{aligned} \tag{2.2}$$

The dilaton/axion fields P, and Q are 1-forms, and their Ansatz is given as follows,

$$\begin{aligned} P &= p_a e^a \\ Q &= q_a e^a \end{aligned} \tag{2.3}$$

while the anti-symmetric tensor 3-form G and self-dual 5-form $F_{(5)}$ are given by

$$\begin{aligned} G &= g_a e^{45a} + i h_a e^{67a} \\ F_{(5)} &= f_a(-e^{0123a} + \varepsilon^{ab} e^{4567b}) \end{aligned} \tag{2.4}$$

Here, f_a, q_a are real, while g_a, h_a, p_a are complex frame vectors on Σ.

2.2 The general solution

In [1], it was shown that the BPS equations imply certain reality conditions on the supersymmetry transformation spinors as well as on the components p_a, q_a, g_a and h_a of the Ansatz. These reality conditions allow every solution of the BPS equations to be mapped, under the $SU(1,1)$ symmetry of Type IIB supergravity, into a solution with vanishing axion field, for which p_a, g_a and h_a are real, and $q_a = 0$. We shall solve the case where these conditions hold. Then applying arbitrary $SU(1,1)$ transformations will produce all solutions with non-vanishing axion field as well, and accounts for 3 parameters for every solution.

[2] We use conventions where the frame indices are denoted by z and $\bar{z}$, the frame metric has non-vanishing components $\delta_{z\bar{z}} = \delta_{\bar{z}z} = 2$, and the orientation on Σ is given by $\varepsilon^{89} = +1$.

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The general solution, with 16 supersymmetries, to Type IIB supergravity on the manifold $AdS_4 \times S_1^2 \times S_2^2 \times \Sigma$ with $SO(2,3) \times SO(3) \times SO(3)$ isometry, is parametrized by two real harmonic functions h_1, h_2 on Σ. The form W of weight $(1,1)$ is ubiquitous in the solution, and is defined by

$$W \equiv \partial_w h_1 \partial_{\bar w} h_2 + \partial_w h_2 \partial_{\bar w} h_1 \tag{2.5}$$

The form W is the inner product between the vector fields $\partial_w h_1$ and $\partial_w h_2$, so that W will vanish when these fields are orthogonal to one another.

The solution for the dilaton field $\Phi = 2\phi$ has the following form,[3]

$$e^{4\phi} = \frac{2h_1 h_2 |\partial_w h_2|^2 - h_2^2 W}{2h_1 h_2 |\partial_w h_1|^2 - h_1^2 W} \tag{2.6}$$

while the solution for the conformal factor ρ^2 in the Σ-metric is given by

$$\rho^8 = \frac{W^2}{h_1^3 h_2^3} \Big(2h_1 |\partial_w h_2|^2 - h_2 W \Big) \Big(2h_2 |\partial_w h_1|^2 - h_1 W \Big) \tag{2.7}$$

The functions f_1, f_2, and f_4 entering the ten-dimensional metric (2.1) are most succinctly expressed by leaving some of the ϕ- and ρ-dependence manifest in the solution,

$$\begin{aligned} \rho\, f_1 &= -2\nu \,\mathrm{Re} \Big(e^{-2\phi} |\partial_w h_2|^2 - e^{2\phi} |\partial_w h_1|^2 - iW \Big)^{\frac{1}{2}} \\ \rho\, f_2 &= -2 \,\mathrm{Im} \Big(e^{-2\phi} |\partial_w h_2|^2 - e^{2\phi} |\partial_w h_1|^2 - iW \Big)^{\frac{1}{2}} \\ \rho\, f_4 &= \Big| e^{-\phi} \partial_w h_2 - i\, e^{\phi} \partial_w h_1 \Big| + \Big| e^{-\phi} \partial_w h_2 + i\, e^{\phi} \partial_w h_1 \Big| \end{aligned} \tag{2.8}$$

Formulas direcly for the metric factors f_1, f_2 and f_4 were derived in appendix E of [1]. Of interest to us in the present paper will be the following combinations, recorded for $W \leq 0$,

$$\begin{aligned} f_1^2 f_4^2 &= 4 e^{+2\phi} h_1^2 \\ f_2^2 f_4^2 &= 4 e^{-2\phi} h_2^2 \end{aligned} \tag{2.9}$$

The expression for $W > 0$ will not be needed in this paper.

Finally, the expressions for the 3-form components g_a, h_a are most usefully presented in terms of the $B_{(2)}$ gauge potential, as this will allow us to directly compute 3-form fluxes and charges. The values for the field strength $F_{(3)} = dB_{(2)}$ may be found in [1], but will not be needed here. We have,

$$B_{(2)} = b_1\, \hat e^{45} + i b_2\, \hat e^{67} \tag{2.10}$$

Here, $\hat e^{45}$ and $\hat e^{67}$ are the volume forms for the spheres S_1^2 and S_2^2 with unit radii. The functions b_1 and b_2 are given by

$$\begin{aligned} b_1 &= +2\tilde h_2 + 2i h_1 h_2 \frac{\partial_w h_1 \partial_{\bar w} h_2 - \partial_{\bar w} h_1 \partial_w h_2}{2h_2 |\partial_w h_1|^2 - h_1 W} \\ b_2 &= -2\tilde h_1 + 2i h_1 h_2 \frac{\partial_w h_1 \partial_{\bar w} h_2 - \partial_{\bar w} h_1 \partial_w h_2}{2h_1 |\partial_w h_2|^2 - h_2 W} \end{aligned} \tag{2.11}$$

[3]For notational convenience, we use $\phi = \Phi/2$ to represent the dilaton, in accord with [1].

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where $\tilde{h}_1$ and $\tilde{h}_2$ are the real harmonic functions conjugate to the harmonic functions h_1 and h_2 respectively. As such, they satisfy the conjugation relations,

$$\begin{aligned} \partial_w h_1 &= -i\partial_w \tilde{h}_1 \\ \partial_w h_2 &= -i\partial_w \tilde{h}_2 \end{aligned} \tag{2.12}$$

These solutions are local in the sense that the range of the local complex coordinate $w, \bar{w}$, and thus the surface Σ, remains to be specified globally.

2.3 The $AdS_5 \times S^5$ and Janus solutions

A 2-parameter family of solutions is obtained from [1],

$$\begin{aligned} h_1 &= 2e^{-\phi_+}\mathrm{Im}\,(e^w) - 2e^{-\phi_-}\mathrm{Im}\left(e^{-w}\right) \\ h_2 &= 2e^{+\phi_+}\mathrm{Re}\,(e^w) + 2e^{+\phi_-}\mathrm{Re}\left(e^{-w}\right) \end{aligned} \tag{2.13}$$

on the infinite strip,

$$\Sigma = \left\{ w \in \mathbf{C};\ 0 \leq \mathrm{Im}(w) \leq \frac{\pi}{2} \right\} \tag{2.14}$$

The dilaton ϕ tends to the constants $\phi_\pm$ as $\mathrm{Re}(w) \to \pm\infty$. When $\phi_+ = \phi_-$, this is just the $AdS_5 \times S^5$ solution, while for $\phi_+ \neq \phi_-$, it is the Janus solution with 16 supersymmetries, predicted in [3] on the basis of its AdS/CFT dual interface super Yang-Mills theory, and calculated explicitly, including the expressions of $\phi, \rho, f_1, f_2, f_4, g_a, h_a, f_a$, in [1].

3. Regularity and topology conditions

To extend the general local solution, given in terms of harmonic functions h_1 and h_2, to a global non-singular solution, we need to specify further global conformal data, namely the domain Σ on which h_1 and h_2 are harmonic. Choosing generic Σ, h_1, and h_2 will generally lead to singularities in the dilaton ϕ and other functions which specify the Ansatz, and thus to singular solutions to Type IIB supergravity. Regularity will require interrelations between Σ, h_1 and h_2, which we shall now exhibit.

3.1 Basic regularity conditions

We shall adopt the following general regularity conditions on all the solutions considered in this paper. The dilaton ϕ and the metric functions ρ^2, f_1, f_2, and f_4 are

(R1) *non-singular in the interior of* Σ;

(R2) *non-singular on the boundary* $\partial\Sigma$, *except possibly at isolated points.*

The $AdS_5 \times S^5$ asymptotic region is one example of an isolated point in $\partial\Sigma$ where f_4 diverges. But the nature of this singularity on $\partial\Sigma$ is well-understood; it corresponds to a regular 10-dimensional geometry, and thus must be allowed. The regularity conditions (R1) and (R2) will also allow for the singularities of probe D5 and/or NS5 branes, for which f_1, f_2, and f_4 all diverge at isolated points on Σ. The fact that local solutions which have poles in the interior of Σ lead to singular geometries is demonstrated for a generic class of singularities in appendix A.

3.2 Topology conditions

In this paper, we shall restrict attention to solutions whose boundary is locally that of $AdS_5 \times S^5$. Poles in ∂h_1 and ∂h_2, located at isolated points on the boundary $\partial\Sigma$ will correspond to asymptotic throat regions, as in the $AdS_5 \times S^5$ and supersymmetric Janus solutions. The remaining part of the boundary consists of open segments. We shall require that these open segments of $\partial\Sigma$ correspond to regular interior points (and not to boundary points) of the full 10-dimensional Type IIB solution. This can be achieved by requiring that throughout each such segment, one S^2, or the other S^2, but not both, shrink to zero size. This topological condition already holds on the $AdS_5 \times S^5$ and Janus solutions, where it guarantees that the spheres S_1^2 and S_2^2 shrink to zero size in a manner precisely needed to recover the overall S^5 topology.

To investigate this topology, it is useful to consider the following combinations of the radii f_1 and f_2, readily derived from (2.7),

$$\rho^4 f_1^2 f_2^2 = 4W^2 \tag{3.1}$$

As a result of our earlier topology condition, the product $f_1 f_2$ must vanish along the entire boundary $\partial\Sigma$. Using the regularity condition (R2) that ρ^2 is non-singular on $\partial\Sigma$, except possibly at isolated points, as well as equations (3.1), it is clear that the vanishing of $f_1 f_2$ on $\partial\Sigma$ implies,

$$W(w,\bar{w}) = 0 \qquad \text{for all} \qquad w \in \partial\Sigma \tag{3.2}$$

For the Janus solution of (2.13), Σ of (2.14) consist of a strip $w \in \mathbf{R} \oplus i[0,\pi/2]$, and f_1 vanishes at 0 while f_2 vanishes at $\pi/2$. Furthermore, we have $W = -4\,\mathrm{ch}\,(\phi_+ - \phi_-)\sin(2\,\mathrm{Im}(w))$, which indeed vanishes throughout the boundary $\partial\Sigma$ of Σ in (2.14).

We shall also require that $\partial\Sigma$ have only a single connected component. From this condition, it follows that $f_1 f_2$ and W cannot vanish in the interior of Σ, except at isolated points. By continuity, the sign of W must remain constant throughout the interior of Σ. If it were not, there would arise a curve on the inside of Σ where $W = 0$, which would contradict the requirement of a single boundary component. Since we seek families of solutions connected to $AdS_5 \times S^5$, for which $W < 0$ inside Σ, we shall require $W < 0$ on the inside of Σ, except at isolated points where W may vanish.

3.3 Interior zeros

Negativity of W greatly restricts the allowed zeros of the Abelian differentials ∂h_1 and ∂h_2. We shall now argue that all zeros of ∂h_1 and ∂h_2 in the interior of Σ must be common. For example, if ∂h_1 has a zero of order m at an interior point $w_0 \in \Sigma$, then ∂h_2 must have a zero of order precisely m at w_0 as well.

To show this, choose local coordinates $v, \bar{v}$ around w_0 so that $v = 0$ at w_0, and

$$\begin{aligned} \partial h_1 &= \partial_v h_1 dv = v^{m_1} dv + \mathcal{O}(v^{m_1+1} dv) \\ \partial h_2 &= \partial_v h_2 dv = c v^{m_2} dv + \mathcal{O}(v^{m_2+1} dv) \qquad\qquad c \neq 0 \end{aligned} \tag{3.3}$$

Parametrizing v and c by polar coordinates $v = |v|e^{i\theta}$, and $c = |c|e^{i\gamma}$, we readily evaluate

$$W = 2|c||v|^{m_1+m_2} \cos\Big((m_1 - m_2)\theta - \gamma\Big) + \mathcal{O}\left(|v|^{m_1+m_2+1}\right) \tag{3.4}$$

Since w_0 is an interior point, the range of θ is over the full circle, $0 \leq \theta \leq 2\pi$. The only manner in which we can have $W < 0$ for sufficiently small $|v|$ and all $\theta \in [0, 2\pi]$ is if $m_1 = m_2$, which proves our earlier assertion. It then suffices to require $\cos\gamma < 0$.

3.4 Boundary conditions on h_1 and h_2

Using (2.9), sharp boundary conditions on h_1 and h_2 may be obtained. First, we argue that the AdS_4-radius f_4 cannot vanish. If follows from equation (6.26) of [1] that, if f_4 vanished, then we also must have $f_1 = f_2 = 0$, resulting in an unphysical singularity of the 10-dimensional Type IIB geometry. In particular, in the $AdS_5 \times S^5$ and Janus solutions, the function f_4 remains bounded away from 0. Combining the facts that $f_4 \neq 0$, and that ϕ is non-singular (except perhaps at isolated points) with equations (2.9), it is clear that f_1 vanishes if and only if $h_1 = 0$, while f_2 vanishes if and only if $h_2 = 0$.

Next, we shall show that, *if $h_2 = 0$ in an open neighborhood $\mathcal{U}_0 \subset \partial\Sigma$, then $h_1 \neq 0$ in $\mathcal{U}_0$, except possibly at isolated points, and h_1 satisfies Neumann boundary conditions in $\mathcal{U}_0$.* The same statement, but with the roles of h_1 and h_2 interchanged, also holds.

To prove that h_1 satisfies Neumann boundary conditions throughout $\mathcal{U}_0$, we use the fact that h_2 is harmonic in Σ, and vanishes on $\mathcal{U}_0$, to choose conformal coordinates $w = x + iy$ (with x, y real), such that $h_2 = y$ in an open neighborhood $\mathcal{S}_0 \subset \Sigma$ which contains $\mathcal{U}_0$. In terms of this coordinate w on $\mathcal{S}_0$, W takes the following form,

$$W = -\mathrm{Im}(\partial_w h_1) = -\partial_y h_1 \tag{3.5}$$

The vanishing of W on the boundary $\partial\Sigma$, as required by (3.2), implies that $\partial_y h_1 = 0$ for all points in $\mathcal{U}_0$, i.e. h_1 satisfies Neumann boundary conditions on $\mathcal{U}_0$.

To prove that $h_1 \neq 0$ in $\mathcal{U}_0$, except possibly at isolated points, we make use of the expression for the dilaton field (2.6) on our solutions. Each term in the numerator and denominator of (2.6) vanishes on $\mathcal{U}_0$, either because $W = 0$, or because $h_2 = 0$, or both. Since with our choice of coordinates w we have $\partial_w h_2 = -i/2$, which is non-vanishing, the first term in the numerator of (2.6) dominates the second. Neglecting the second term allows us to simplify by a factor of h_1 and we are left with,

$$e^{4\phi} \sim \frac{1}{4|\partial_w h_1|^2 + 2y^{-1} h_1 \partial_y h_1} \tag{3.6}$$

If h_1 were to vanish throughout an open set $\mathcal{U}_0' \subset \mathcal{U}_0$, then $|\partial_w h_1|^2$ must also vanish there, since then we would have both $\partial_x h_1 = 0$ and $\partial_y h_1 = 0$. Furthermore, since $\partial_y h_1 = 0$ in $\mathcal{U}_0'$, it follows that, if h_1 also vanishes there, we actually must have $y^{-1}h_1 \to 0$ as $y \to 0$. As a result, the denominator has to vanish as $y \to 0$, and the dilaton would be singular throughout the open set $\mathcal{U}_0'$, which contradicts assumption (R2) that the dilaton must be non-singular on $\partial\Sigma$, except possibly at isolated points. Thus, we must have $h_1 \neq 0$ on $\mathcal{U}_0$, except possibly at isolated points. This completes the proof of our above statement.

In summary, we have a remarkable conclusion. The boundary $\partial\Sigma$ is partitioned into two open sets $\partial\Sigma_+$ and $\partial\Sigma_-$. The closures of $\partial\Sigma_+$ and $\partial\Sigma_-$ intersect at isolated points, and their union is $\partial\Sigma$. The following boundary conditions on the functions h_1 and h_2 now hold,

$$
\begin{array}{lll}
\text{on } \partial\Sigma_+ & f_1 = h_1 = 0 & h_1 = \text{vanishing Dirichlet} \\
 & \partial_n h_2 = 0 & h_2 = \text{Neumann} \\
 & & \\
\text{on } \partial\Sigma_- & f_2 = h_2 = 0 & h_2 = \text{vanishing Dirichlet} \\
 & \partial_n h_1 = 0 & h_1 = \text{Neumann}
\end{array}
\tag{3.7}
$$

where ∂_n denotes the derivative normal to the boundary $\partial\Sigma$.

3.5 Summary of all regularity and topology conditions

Having analyzed the behavior of the harmonic functions h_1 and h_2 on the boundary $\partial\Sigma$, it remains to determine their allowed behavior on the inside of Σ. Given that f_4 cannot vanish in Σ, as argued in the first paragraph of subsection 3.4, it follows from (2.9) that h_1 and h_2 cannot vanish on the inside of Σ. Indeed a line or domain of zeros would force f_1 or f_2 to vanish in the interior of Σ, which is inconsistent with the assumption of subsection 3.2 that $\partial\Sigma$ consists of only a single connected component. For the Janus solution in (2.13), we have $h_1, h_2 > 0$ in the interior of Σ. Since our solutions will be connected to $AdS_5 \times S^5$, we require $h_1, h_2 > 0$ on the inside of Σ throughout.

In summary, we have the following regularity conditions, in addition to (R1) and (R2),

(R3) On $\partial\Sigma_+$, h_1 and h_2 obey respectively Dirichlet and Neumann boundary conditions; On $\partial\Sigma_-$, h_1 and h_2 obey respectively Neumann and Dirichlet boundary conditions; Note that these two conditions together imply $W = 0$ throughout $\partial\Sigma$.

(R4) All zeros of ∂h_1 and ∂h_2 on the inside of Σ must be common;

(R5) $W < 0$ on the inside of Σ, except possibly at isolated points, where $W = 0$;

(R6) $h_1 > 0$ and $h_2 > 0$ on the inside of Σ;

(R7) All Dirichlet boundary conditions must be vanishing, as given in (3.7).

Remarkably, this combination of regularity and topology assumptions, leads to boundary conditions akin to those of 2-dimensional electro-statics, with Dirichlet and Neumann components corresponding respectively to perfect conductor and perfect insulator.

4. The hyperelliptic Ansatz

We shall now present a construction for the harmonic functions h_1 and h_2, in terms of hyperelliptic Riemann surfaces of genus g, which automatically solve the conditions (R1-R5) of section 3.5. Conditions (R6) and (R7) are more subtle, and will lead to certain relations on the parameters, which we shall derive explicitly. The genus 0 case reduces to the Janus and $AdS_5 \times S^5$ solutions, which are completely non-singular. For general genus g, the Ansatz has $4g+6$ parameters (including one for the overall shift of the dilaton one for the overall scale of the 10-dimensional metric, and 3 for the SU(1,1) rotation parameters to the general solution with non-vanishing axion.) In the next section, all regularity conditions will be solved analytically for the case of genus 1, and we shall prove that, in a certain range of these parameters, the full geometry of the solution is non-singular.

4.1 The Janus solution re-expressed on the lower half-plane

To begin construction of the hyperelliptic Ansatz, we map the Janus and $AdS_5 \times S^5$ solutions to the lower half-plane Σ, on which the complex coordinates will be denoted by $u, \bar{u}$ with $\mathrm{Im}(u) \leq 0$. This may be done with the help of the exponential mapping,

$$-\sqrt{u} = e^{-w-(\phi_+-\phi_-)/2} \qquad\qquad 0 \leq \mathrm{Im}(w) \leq \frac{\pi}{2} \tag{4.1}$$

where $\mathrm{Re}(w), \mathrm{Re}(u)$ take values through $\mathbf{R}$. Notice that $\mathrm{Re}(w) = -\infty$ and $\mathrm{Re}(w) = +\infty$ are respectively mapped to $u = \infty$ and $u = 0$. The harmonic functions h_1, h_2 and their differentials $\partial h_1, \partial h_2$ are given by,

$$\begin{aligned} h_1(u) &= i\,\frac{r-u}{\sqrt{u}} + c.c. & \qquad \partial h_1 &= -\frac{i}{2}\,\frac{u+r}{\sqrt{u}^3}\,du \\ h_2(u) &= -\frac{1+u}{\sqrt{u}} + c.c. & \qquad \partial h_2 &= -\frac{1}{2}\,\frac{u-1}{\sqrt{u}^3}\,du \end{aligned} \tag{4.2}$$

Here, we use the shorthand $r = e^{2\phi_- - 2\phi_+}$ and, to simplify the form of the differentials, we have omitted overall multiplicative constants in h_1 and h_2 whose effect is to shift the dilaton by an overall constant; these factors may be easily restored. From the differentials, we see that ∂h_1 is real along the negative real axis and imaginary along the positive real axis, and thus respectively obeys Neumann and Dirichlet boundary conditions along these segments. For ∂h_2, the situation is reversed, as depicted in figure 2. Thus, the solution (4.2) satisfies the conditions (3.7) with $\partial\Sigma_+ =]0, +\infty[$ and $\partial\Sigma_- =]-\infty, 0[$. It was shown in [1] that the solution is everywhere non-singular, so that all conditions (R1-R7) are in fact satisfied.

4.2 The hyperelliptic Ansatz

The Janus solution may be generalized by having, instead of just a single vanishing Dirichlet (D) segment $]0, +\infty[$, and a single Neumann (N) segment $]-\infty, 0[$ for h_1, and opposite boundary conditions for h_2, a larger number of such segments. This may be achieved by arranging the differentials ∂h_1 and ∂h_2 to have alternating real and purely imaginary values over several segments along $\partial\Sigma$. Since topology requires the boundary $\partial\Sigma$ to have only a single connected component, we may conformally map $\partial\Sigma$ to the entire real axis, and the interior of Σ to the lower half-plane. The hyperelliptic Ansatz is obtained by having $2g+2$ instead of 2 separate segments of alternating D and N boundary conditions on the real axis $\partial\Sigma$.

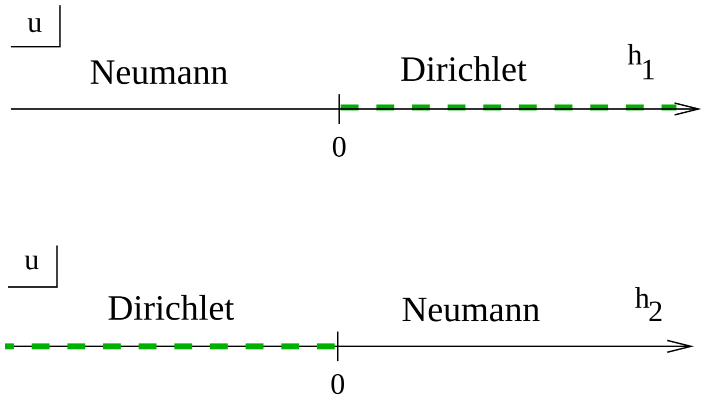

Figure 2: The cut plane for h_1 and h_2 of the Janus solution.

Multiple alternating D and N boundary conditions may be obtained in terms of $2g+2$ branch points on the real line, $e_1, e_2, \cdots, e_{2g+1}$, and $e_{2g+2} = \infty$, and the polynomial

$$s^2 = (u - e_1) \prod_{k=1}^{g} (u - e_{2k})(u - e_{2k+1}) \tag{4.3}$$

Here, we have used the SL$(2, \mathbf{R})$ symmetry on the lower complex half-plane to fix one branch point at ∞; two further branch points may be chosen at arbitrary points as well, using SL$(2, \mathbf{R})$. The algebraic equation (4.3) defines a hyperelliptic surface of genus g. Its $2g-1$ real moduli for $g \geq 1$, may be parametrized by the $2g-1$ remaining branch points. It will be convenient to prescribe the following definite ordering for the branch points,

$$e_{2g+1} < e_{2g} < e_{2g-1} < \cdots < e_3 < e_2 < e_1 \tag{4.4}$$

The collapse of any pair of consecutive branch points will produce a non-separating degeneration to a hyperelliptic surface of genus $g-1$.

To produce a suitable generalization which satisfies the regularity conditions (R1) and (R2), we shall assume that the singularities of the differentials ∂h_1 and ∂h_2 at each branch point are no worse than those of the $AdS_5 \times S^5$ solution, i.e. of the form $du/(u-e_i)^{3/2}$. The differentials then have at most double poles at the branch points. A double pole at $u = \infty$ translates to the asymptotic behavior $du/\sqrt{u}$ as $u \to \infty$. Poles on the inside of Σ would lead to singular solutions, as shown in appendix A, and are thus excluded by (R1) and (R2).[4] Combining these requirements, we have the following form of the differentials,

$$\begin{aligned} \partial h_1 &= -i \frac{P_1(u)}{s(u)^3} du \\ \partial h_2 &= -\frac{P_2(u)}{s(u)^3} du \end{aligned} \tag{4.5}$$

[4]Certain types of poles on the real axis will lead to mildly singular space-time solutions, which are of the type of D5 or NS5 branes in the probe limit. These solutions will in fact be recovered as limits of regular hyperelliptic solutions, as will be demonstrated in section 7.

where $P_1(u)$ and $P_2(u)$ are polynomials of degree $3g+1$ in u. The overall $-$ sign for both ∂h_1 and ∂h_2 has been introduced later convenience.

To realize condition (R3) on the differentials, ∂h_1 and ∂h_2 need to alternate between taking real and purely imaginary values on the real axis. Along the real axis, the sign of s^2 behaves as follows,

$$
\begin{aligned}
s^2 > 0 \quad u \in \partial\Sigma_+ &\equiv \;]e_1, +\infty[\cup \bigcup_{i=1}^{g}]e_{2i+1}, e_{2i}[\\
s^2 < 0 \quad u \in \partial\Sigma_- &\equiv \;]-\infty, e_{2g+1}[\cup \bigcup_{i=1}^{g}]e_{2i}, e_{2i-1}[
\end{aligned}
\tag{4.6}
$$

so that the denominator $s(u)^3$ alternates between being real and purely imaginary on the real axis. Thus, to realize (R3), we must require that $P_1(u)$ and $P_2(u)$ either both be real, or both be purely imaginary on the real axis. Without loss of generality, we may choose both P_1 and P_2 to be real, i.e. to be polynomials with all real coefficients. The boundary conditions (R3) are then automatically satisfied, and we have

$$
\begin{array}{llll}
u \in \partial\Sigma_+ & \partial_u h_1 = \text{imaginary} & \text{Dirichlet} & \\
& \partial_u h_2 = \text{real} & \text{Neumann} & \\
u \in \partial\Sigma_- & \partial_u h_1 = \text{real} & \text{Neumann} & \\
& \partial_u h_2 = \text{imaginary} & \text{Dirichlet} & (4.7)
\end{array}
$$

Note that, given the alternating structure of the D and N boundary conditions, it would have been unnatural for our solutions to use either h_1 or h_2 as the real or imaginary part of a global system of conformal coordinates. on Σ.

4.3 Structure of the complex zeros

Condition (R4) of subsection 3.5 requires all the interior zeros, for which $\mathrm{Im}(u) < 0$, of ∂h_1 and ∂h_2 to be common. The polynomials P_1 and P_2 have real coefficients, so that their zeros are either real or come in complex pairs. These complex zeros must be common between P_1 and P_2 in view of (R4), so that we have the decomposition, $P_1(u) = P(u)Q_1(u)$ and $P_2(u) = P(u)Q_1(u)$, where $P(u)$ is a real polynomial whose roots are all complex and $Q_1(u)$ and $Q_2(u)$ are polynomials with only real roots. Thus, we obtain the more specific forms,

$$
\begin{aligned}
\partial h_1 &= -i\,\frac{P(u)Q_1(u)}{s(u)^3}du \\
\partial h_2 &= -\frac{P(u)Q_2(u)}{s(u)^3}du
\end{aligned}
\tag{4.8}
$$

We shall assume that the roots of Q_1 are distinct; coincident roots may be attained as a limit later. We shall parametrize and order the roots of these polynomials as follows,

$$\begin{aligned}
P(u) &= \prod_{a=1}^{p}(u-u_a)(u-\bar u_a) && \mathrm{Im}(u_a)<0 \\
Q_1(u) &= \prod_{b=1}^{q}(u-\alpha_b) && \alpha_q<\alpha_{q-1}<\cdots<\alpha_2<\alpha_1 \\
Q_2(u) &= \prod_{b=1}^{q}(u-\beta_b) && \beta_q<\beta_{q-1}<\cdots<\beta_2<\beta_1
\end{aligned} \tag{4.9}$$

Here, $\alpha_b, \beta_b \in \mathbf{R}$ and $3g+1 = 2p+q$ in view of the fact that P_1 and P_2 have degree $3g+1$.

4.4 Negativity of W

Condition (R5) of subsection 3.5 requires that $W<0$ for all $\mathrm{Im}(u)<0$, except possibly at isolated points, where we may have $W=0$. Clearly, at the common complex zeros u_a both differentials ∂h_1 and ∂h_2 and thus W all vanish. The points u_a are, of course, isolated points, and these zeros of W fit with the terms of condition (R5). To investigate the condition $W<0$ elsewhere, we recast W in terms of the polynomials P, Q_1, Q_2,

$$W = -\frac{|P(u)Q_1(u)|^2}{|s|^6}\left(i\frac{Q_2(u)}{Q_1(u)} - i\frac{Q_2(\bar u)}{Q_1(\bar u)}\right) \tag{4.10}$$

Since all the zeros of Q_1 are real and simple, and Q_1 and Q_2 are of the same degree q and start with the same highest monomial u^q, the ratio Q_2/Q_1 may be decomposed as follows,

$$Q(u) \equiv \frac{Q_2(u)}{Q_1(u)} = 1 - \sum_{b=1}^{q}\frac{\gamma_b}{u-\alpha_b} \tag{4.11}$$

where the residues γ_b are real. As a result, W takes the form,

$$W = -i(u-\bar u)\frac{|P(u)Q_1(u)|^2}{|s|^6}\sum_{b=1}^{q}\frac{\gamma_b}{|u-\alpha_b|^2} \tag{4.12}$$

The requirement $W \leq 0$ in the lower half-plane is equivalent to

$$\gamma_b \geq 0 \qquad\qquad b=1,\cdots,q \tag{4.13}$$

Notice that if $\gamma_b=0$ then α_b is a zero common to Q_1 and Q_2. We begin by assuming that $\gamma_b>0$ for all $b=1,\cdots,q$, the alternative may be attained as a limiting case hereof. The condition $\gamma_b>0$ is equivalent to an ordering condition between the zeros of Q_1 and Q_2, as shown in the Lemma below.

Lemma.

(a) For $\gamma_b>0$, $b=1,\cdots,q$, the zeros β_b of Q_2 and the zeros α_b of Q_1, alternate,

$$\alpha_q < \beta_q < \alpha_{q-1} < \beta_{q-1} < \cdots < \alpha_2 < \beta_2 < \alpha_1 < \beta_1 \tag{4.14}$$

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(b) Vice-versa, if $Q_1(x)$ and $Q_2(x)$ have highest monomial x^q and real zeros which alternate as in (4.14), then $Q(x) = Q_2(x)/Q_1(x)$ admits a pole decomposition (4.11) with $\gamma_b > 0$.

To prove part (a), we analyze the behavior of $Q(x)$ for real x. Since all $\gamma_b > 0$, none of the α_b is a zero of Q_2 or Q. On the other hand, we have,

$$Q'(x) = \sum_{b=1}^{q} \frac{\gamma_b}{(x-\alpha_b)^2} > 0 \tag{4.15}$$

so that $Q(x)$ is monotonically increasing between any two consecutive poles. Indeed, the behavior close to a pole α_b is

$$Q(x) \sim \frac{-\gamma_b}{x-\alpha_b} \tag{4.16}$$

so that $Q(x)$ tends to $+\infty$ on the left and to $-\infty$ on the right of every pole. Thus, within an interval $[\alpha_{b+1}, \alpha_b]$ between any two consecutive poles, $Q(x)$ goes from $-\infty$ (at α_{b+1}) to $+\infty$ (at α_b). Therefore, it must attain the value zero once and only once in that interval. Therefore, in the interval $[\alpha_q, \alpha_1]$ the function $Q_2(x)$ has precisely $q-1$ zeros. The single remaining zero of $Q_2(x)$ is on the right of this interval, which proves part (a) of the Lemma.

To prove part (b) of the Lemma, we simply notice that

$$\gamma_b = -\frac{Q_2(\alpha_b)}{Q_1'(\alpha_b)} = -(\alpha_b - \beta_b) \prod_{a \neq b} \frac{(\alpha_b - \beta_a)}{(\alpha_b - \alpha_a)} \tag{4.17}$$

By inspection of (4.14), it is clear that, given b, there are just as many $\alpha_a > \alpha_b$ as there are $\beta_a > \alpha_b$, so that the product of ratios in the above formula is always positive. Since we also have $-(\alpha_b - \beta_b) > 0$, it follows that $\gamma_b > 0$, and this completes the proof of the Lemma.

We conclude that the condition $W < 0$ throughout the lower half-plane, for polynomials Q_1 with zeros $\alpha_1, \cdots, \alpha_q$, and Q_2, with zeros $\beta_1, \cdots, \beta_q$, is equivalent to the alternation of their roots in (4.14), which thus gives a complete description and parametrization of $W < 0$. The limit of coincident zeros $\alpha_c = \alpha_{c+1}$ for some $1 \leq c \leq q-1$ forces $\beta_c = \alpha_c$.

4.5 Obtaining the harmonic functions h_1, h_2

Thus far, our attention has focussed on the differentials ∂h_1 and ∂h_2. The full supergravity solution, of course, also involves the harmonic functions h_1 and h_2 themselves, as will the final two regularity conditions (R6) and (R7) of section subsection 3.5. From the differentials ∂h_1 and ∂h_2, the harmonic functions are obtained as hyperelliptic Abelian integrals with double poles at each branch point. It will be important to simplify these Abelian integrals and express them in terms of Abelian integrals with a pole only at infinity. This is carried out in this subsection.

The functions h_1 and h_2 will have simple poles at the branch points. It will be convenient to expose these poles before analyzing the boundary conditions on h_1 and h_2. A convenient basis for meromorphic functions with a single pole at branch point e_i is given by

$$\frac{s(u)}{u - e_i} \tag{4.18}$$

A useful formula is obtained by taking the differential of these basis functions. Recall that the differential $du/s(u)$ is holomorphic, with a single zero at ∞, which is of order $2g-2$. To take the differentials of the functions $s(u)/(u-e_i)$, we introduce the following notation,

$$\begin{aligned} s(u)^2 &= (u-e_i)F_i(u) \\ F_i(u) &= (u-e_i)\Big(G_i(u) + F_i'(u)\Big) + E_i \\ E_i &\equiv F_i(e_i) \end{aligned} \tag{4.19}$$

By construction, we have $G_i(e_i) = 0$. The polynomials $s(u)^2, F_i(u), G_i(u)$ are of respective degrees $2g+1, 2g$, and $2g-1$, while E_i do not depend on u. Using (4.19), we may recast the double poles in terms of total derivatives of simple poles, so we shall use this formula in the following manner,

$$\frac{E_i\, du}{(u-e_i)s(u)} = -G_i(u)\frac{du}{s(u)} - 2d\left(\frac{s(u)}{u-e_i}\right) \tag{4.20}$$

To apply this formula to the differentials ∂h_1, ∂h_2, we decompose the ratios $P(u)Q_{1,2}(u)/s(u)^2$ onto the double poles at the branch points, and recast the differentials as follows,

$$\begin{aligned} \partial h_1 &= -i\left(R_1(u) - \sum_{i=1}^{2g+1}\frac{E_iA_i}{u-e_i}\right)\frac{du}{s(u)} \\ \partial h_2 &= -\left(R_2(u) - \sum_{i=1}^{2g+1}\frac{E_iB_i}{u-e_i}\right)\frac{du}{s(u)} \end{aligned} \tag{4.21}$$

where $R_1(u)$ and $R_2(u)$ are polynomials in u of degree g, with highest degree monomial u^g. The polynomials $R_1(u)$ and $R_2(u)$ may be evaluated using an expansion for large u. By inspection, the Abelian differentials $R_{1,2}(u)du/s(u)$ are meromorphic with a double pole at ∞, and $2g$ zeros. Identifying residues at $u = e_i$ gives,

$$\begin{aligned} A_i &= -P(e_i)Q_1(e_i)\,(E_i)^{-2} \\ B_i &= -P(e_i)Q_2(e_i)\,(E_i)^{-2} \end{aligned} \tag{4.22}$$

Expressing the double poles in terms of differentials of simple poles, using (4.20), gives

$$\begin{aligned} \partial h_1 &= -ip_1(u)\frac{du}{s(u)} - idq_1(u) \\ \partial h_2 &= -p_2(u)\frac{du}{s(u)} - dq_2(u) \end{aligned} \tag{4.23}$$

where

$$
\begin{aligned}
q_1(u) &= 2\sum_{i=1}^{2g+1}\frac{s(u)A_i}{u-e_i} & \qquad p_1(u) &= R_1(u)+\sum_{i=1}^{2g+1}G_i(u)A_i \\
q_2(u) &= 2\sum_{i=1}^{2g+1}\frac{s(u)B_i}{u-e_i} & \qquad p_2(u) &= R_2(u)+\sum_{i=1}^{2g+1}G_i(u)B_i
\end{aligned}
\tag{4.24}
$$

Here, p_1 and p_2 are real polynomials of u of degree $2g-1$, and q_1 and q_2 are real algebraic functions of u. As a result, we have explicit formulas for the functions h_1 and h_2 themselves,

$$
\begin{aligned}
h_1(u) &= h_1^{(0)} + 2\mathrm{Im}\Big(q_1(u)\Big) + 2\mathrm{Im}\left(\int_{u_1}^{u}\frac{p_1(u)du}{s(u)}\right) \\
h_2(u) &= h_2^{(0)} - 2\mathrm{Re}\Big(q_2(u)\Big) - 2\mathrm{Re}\left(\int_{u_2}^{u}\frac{p_2(u)du}{s(u)}\right)
\end{aligned}
\tag{4.25}
$$

where $h_1^{(0)}$ and $h_2^{(0)}$ are real integration constants and u_1, u_2 are the origins of integration on the real axis. The remaining Abelian integrals are smooth functions everywhere, including at the branch points e_i, except possibly at the branch point at $\pm\infty$.

4.6 Positivity of h_1, h_2 near the branch points

Condition (R6) of 3.5 requires that $h_1 > 0$ and $h_2 > 0$ for all $\mathrm{Im}(u) < 0$. We begin by ensuring their positivity at the branch points e_i, where the behavior of the simple pole dominates. Using (4.25) near a branch point $u \sim e_i$, we have

$$
\begin{aligned}
h_1(u) &= +2A_i\,\mathrm{Im}\left(\frac{s(u)}{u-e_i}\right) + \mathcal{O}(1) \\
h_2(u) &= -2B_i\,\mathrm{Re}\left(\frac{s(u)}{u-e_i}\right) + \mathcal{O}(1)
\end{aligned}
\tag{4.26}
$$

Some care is needed in analyzing this condition. Starting from $u = +\infty$, move u along the real axis to the left. On the branch $]e_1, +\infty[$, we define the function $s(u)$ to be negative. (Making the positive choice would flip the signs of both h_1 and h_2, which would be an immaterial change in our set-up.) This fixes the phase of $s(u)$ for all $u \in \mathbf{R}$, as follows,

$$
\begin{aligned}
s(u)/|s(u)| &= -1 & \qquad u &\in\ ]e_1,+\infty[\cup\bigcup_{j=1}^{n_1}]e_{4j+1},e_{4j}[\\
s(u)/|s(u)| &= +i & \qquad u &\in \bigcup_{j=0}^{n_2}]e_{4j+2},e_{4j+1}[\\
s(u)/|s(u)| &= +1 & \qquad u &\in \bigcup_{j=0}^{n_3}]e_{4j+3},e_{4j+2}[\\
s(u)/|s(u)| &= -i & \qquad u &\in \bigcup_{j=0}^{n_4}]e_{4j+4},e_{4j+3}[
\end{aligned}
\tag{4.27}
$$

The upper limits of these unions are given by

$$\begin{aligned} n_1 &= [g/2] \\ n_2 &= [(2g-1)/4] + n_{-\infty} \\ n_3 &= [(g-1)/2] \\ n_4 &= [(2g-3)/2] + 1 - n_{-\infty} \end{aligned} \tag{4.28}$$

where $n_{-\infty} = 1$ when g is even and $n_{-\infty} = 0$ when g is odd, and $e_{2g+2} = -\infty$.

We now turn to evaluating the signs of the pole contributions to h_1 and h_2. Approaching the branch points e_{2j} from the left and the branch points e_{2j-1} from the right yields zero contribution to $h_1(u)$, because the ratio $s(u)/(u-e_i)$ is real there, while approaching the branch points e_{2j} from the right and the branch points e_{2j-1} from the left yields zero contribution to $h_2(u)$, because the ratio $s(u)/(u-e_i)$ is imaginary there.

Non-vanishing contributions are obtained for h_1 when e_{2j} is approached from the right and e_{2j-1} from the left. For h_1, approaching e_{4j+2} from the right and e_{4j+1} from the left gives $s/|s| = +i$; while approaching e_{4j+4} from the right and e_{4j+3} from the left gives $s/|s| = -i$. For h_2, approaching e_{4j+2} from the left and e_{4j+3} from the right gives $s/|s| = +1$; while approaching e_{4j+4} from the left and e_{4j+1} from the right gives $s/|s| = -1$. These values suffice to establish the signs required on A_i and B_i to make $h_1, h_2 > 0$ at the branch points, and we find,

$$\begin{aligned} A_{4j} &< 0 \qquad & B_{4j} &< 0 \\ A_{4j+1} &< 0 \qquad & B_{4j+1} &> 0 \\ A_{4j+2} &> 0 \qquad & B_{4j+2} &> 0 \\ A_{4j+3} &> 0 \qquad & B_{4j+3} &< 0 \end{aligned} \tag{4.29}$$

Using the fact that we always have $P(e_i) > 0$, it is straightforward to translate these conditions into equivalent conditions of the polynomials Q_1 and Q_2, and we find,

$$\begin{aligned} Q_1(e_{4j}) &> 0 \qquad & Q_2(e_{4j}) &> 0 \\ Q_1(e_{4j+1}) &> 0 \qquad & Q_2(e_{4j+1}) &< 0 \\ Q_1(e_{4j+2}) &< 0 \qquad & Q_2(e_{4j+2}) &< 0 \\ Q_1(e_{4j+3}) &< 0 \qquad & Q_2(e_{4j+3}) &> 0 \end{aligned} \tag{4.30}$$

Together with the ordering of the branch points e_i and the zeros of Q_1 and Q_2 these conditions give us the possible relative orderings of these points. For higher genus, there are clearly many combinatorial possibilities.

Of course, having proven that $h_1, h_2 > 0$ near the branch points for the above assignments does not prove that positivity holds throughout the lower half-plane. This result requires first solving the regularity condition (R7) of subsection 3.5, which we do in the next subsection.

4.7 Period relations

Condition (R7) of subsection 3.5, requires that wherever h_1 or h_2 satisfies Dirichlet conditions, it actually must vanish there: all Dirichlet boundary conditions must actually be *vanishing*. Thus, we must require that h_1 and h_2 satisfy vanishing Dirichlet boundary conditions on $\partial\Sigma_+$ and $\partial\Sigma_-$ respectively.

We begin by requiring $h_1 = 0$ on the Dirichlet segment $[e_1, +\infty] \subset \partial\Sigma_+$, and $h_2 = 0$ on the Dirichlet segment $[-\infty, e_{2g+1}] \subset \partial\Sigma_-$. As a result, the integration constants in (4.25) get fixed, and we have[5]

$$\begin{aligned} h_1(u) &= 2\,\mathrm{Im}\Big(q_1(u)\Big) + 2\,\mathrm{Im}\left(\int_{+\infty}^{u} \frac{p_1(u)du}{s(u)}\right) \\ h_2(u) &= -2\,\mathrm{Re}\Big(q_2(u)\Big) - 2\,\mathrm{Re}\left(\int_{-\infty}^{u} \frac{p_2(u)du}{s(u)}\right) \end{aligned} \tag{4.31}$$

Here, we have also used the fact that q_1 is manifestly real on $[e_1, +\infty]$, and q_2 manifestly imaginary on $[-\infty, e_{2g+1}]$. To guarantee that $h_1(u) = 0$ also when $u \in [e_{2j+1}, e_{2j}]$, for $j = 1, \cdots, g$, and $h_2(u) = 0$ for $u \in [e_{2j}, e_{2j-1}]$ for $j = 1, \cdots, g$, it will suffice to have

$$\begin{aligned} h_1(e_{2j}) - h_1(e_{2j-1}) &= 0 \qquad\qquad j = 1, \cdots, g \\ h_2(e_{2j+1}) - h_2(e_{2j}) &= 0 \qquad\qquad j = 1, \cdots, g \end{aligned} \tag{4.32}$$

The contributions from q_1 and q_2 vanish provided the branch points are approached from a suitable direction (the path of integration of the differentials ∂h_1 and ∂h_2 has to be smooth). The remaining conditions (4.32) amount to the following period relations,

$$\begin{aligned} \mathrm{Im}\left(\int_{e_{2j}}^{e_{2j-1}} \frac{p_1(u)du}{s(u)}\right) &= 0 \qquad\qquad j = 1, \cdots, g \\ \mathrm{Re}\left(\int_{e_{2j+1}}^{e_{2j}} \frac{p_2(u)du}{s(u)}\right) &= 0 \qquad\qquad j = 1, \cdots, g \end{aligned} \tag{4.33}$$

giving a system of $2g$ real linear relations on the coefficients of the polynomials p_1 and p_2.

4.8 Positivity of h_1 and h_2 throughout Σ

Combining the results of positivity near the branch points, obtained in (4.30), with the vanishing of h_1 and h_2 on their respective Dirichlet segments, we can now derive a further positivity condition for h_1 and h_2. It stems from the fact that if h_1 and h_2 vanish on their Dirichlet segments and are to be positive in the upper half-plane, then their normal derivatives on the real axis must be negative, namely

$$\begin{aligned} \mathrm{Im}\,(\partial_u h_1)\Big|_{\mathrm{Im}(u)=0} &< 0 \qquad\qquad u \in \partial\Sigma_+ \\ \mathrm{Im}\,(\partial_u h_2)\Big|_{\mathrm{Im}(u)=0} &< 0 \qquad\qquad u \in \partial\Sigma_- \end{aligned} \tag{4.34}$$

[5] To properly define these integrals up to infinity, one should first introduce a cutoff, and then take the limit. No contributions to h_1 or h_2 arise from these segments, however, whether finite or infinite, because we take the real or imaginary parts.

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Using the fact that $P(u) > 0$ for all $\mathrm{Im}(u) = 0$, and the phase values of s along the real axis, given in (4.27), we obtain the following inequalities for Q_1 and Q_2,

$$\begin{aligned}
&Q_1(u) > 0 \qquad && u \in [e_1, +\infty] \cup \bigcup_{j=1}^{n_1} [e_{4j+1}, e_{4j}] \\
&Q_2(u) < 0 \qquad && u \in \bigcup_{j=0}^{n_2} [e_{4j+2}, e_{4j+1}] \\
&Q_1(u) < 0 \qquad && u \in \bigcup_{j=0}^{n_3} [e_{4j+3}, e_{4j+2}] \\
&Q_2(u) > 0 \qquad && u \in \bigcup_{j=0}^{n_4} [e_{4j+4}, e_{4j+3}]
\end{aligned} \tag{4.35}$$

with the same assignments for $n_1, \cdots, n_4$ as in (4.28). These conditions imply the inequalities at the branch points in (4.30), but they are actually stronger, as they imply that neither Q_1 nor Q_2 can have any zeros in their respective Dirichlet segments.

4.9 The general hyperelliptic solution

First, we shall show that the sign conditions on $Q_1(e_i)$ and $Q_2(e_i)$ put a lower bound on the number of real zeros. The existence of a lower bound follows from the fact, in (4.35), that the sign of $Q_1(e_k)$ and $Q_2(e_k)$ alternates in k with periodicity 4. This alternation requires the degrees of the polynomials to satisfy $q = \deg(Q_1) = \deg(Q_2) \geq g$. Using the relation $3g + 1 = 2p + q$ of (4.9), where p is the number of complex zeros in the lower half-plane, it is clear that the bound $q = g$ can never be attained, and we have a more stringent bound,

$$g + 1 \leq q = \deg(Q_1) = \deg(Q_2) \tag{4.36}$$

which in turn implies that the number p of complex zeros u_a obeys $p \leq g$.

Second, we shall use the fact, derived in (4.35), that Q_1 and Q_2 have no zeros on their respective Dirichlet segments, so that,

$$\begin{aligned}
\{\alpha_1, \cdots, \alpha_q\} &\subset \mathcal{U}_- \\
\{\beta_1, \cdots, \beta_q\} &\subset \mathcal{U}_+
\end{aligned} \tag{4.37}$$

as well as the fact that the condition $W < 0$ requires the zeros of Q_1 and Q_2 to alternate as in (4.14). Since each elementary interval $[e_1, +\infty]$ and $[e_{i+1}, e_i]$ for $i = 1, \cdots, 2g - 1$ either contains no α roots or no β roots, we conclude that it cannot contain more than one root (whether α or β). Suppose, for example, that the interval $[e_{4j+2}, e_{4j+1}]$ contained two consecutive α roots as (4.14), then it would also have to contain the β root which lies in between the two α roots and this is not allowed by (4.35). Thus, each elementary interval contains at most one root. This gives an upper bound on the total number of real roots,

$$2q \leq 2g + 2 \tag{4.38}$$

which we simply obtain from counting the total number of elementary intervals, $2g+2$. Combining the lower bound on q in (4.36) and the upper bound on q in (4.38), it is immediate that we must have

$$\begin{aligned} p &= g \\ q &= g+1 \end{aligned} \tag{4.39}$$

which implies a *unique relative ordering of the roots and branch points*, given by

$$\alpha_{g+1} < e_{2g+1} < \beta_{g+1} < e_{2g} < \cdots < \alpha_b < e_{2b-1} < \beta_b < \cdots < e_2 < \alpha_1 < e_1 < \beta_1 \tag{4.40}$$

for $g \geq b \geq 2$. Once this ordering is satisfied, conditions (R5) and (R6) will be obeyed.

4.10 Summary of the hyperelliptic solution

In summary, the genus g hyperelliptic Riemann surface Σ is represented by the lower half-plane, and its boundary $\partial\Sigma$ is represented by the real line. The $2g+2$ branch points $e_1, \cdots, e_{2g+1}, e_{2g+2} = \infty$ are real. The harmonic functions are defined by their differentials, ∂h_1 and ∂h_2, given by

$$\begin{aligned} \partial h_1 &= -i \frac{P(u)Q_1(u)}{s(u)^3} du \\ \partial h_2 &= -\frac{P(u)Q_2(u)}{s(u)^3} du \end{aligned} \tag{4.41}$$

Here, s was given in (4.3); P is a polynomial of degree $2g$ in u, whose zeros come in complex pairs $(u_a, \bar{u}_a)$, with $\mathrm{Im}(u_a) < 0$; and Q_1 and Q_2 are polynomials of degree $g+1$ in u with real roots $\alpha_1, \cdots, \alpha_{g+1}$ and $\beta_1, \cdots, \beta_{g+1}$ respectively. The form of the differentials ∂h_1 and ∂h_2, and the relative ordering (4.40) between the real zeros and the branch points guarantees that conditions (R1-R7) are satisfied provided we impose also the period relations

$$\begin{aligned} \mathrm{Im}\left(\int_{e_{2j}}^{e_{2j-1}} \partial h_1\right) &= 0 \qquad\qquad j = 1, \cdots, g \\ \mathrm{Im}\left(\int_{e_{2j+1}}^{e_{2j}} \partial h_2\right) &= 0 \qquad\qquad j = 1, \cdots, g \end{aligned} \tag{4.42}$$

Given a set of real zeros and branch points with ordering (4.40), the $2g$ period relations determine the g complex zeros u_a.

Solutions of these period relations may not exist for given moduli of Σ, and all values of the real roots α and β consistent with (4.40). In the section 5, the allowed parameter space will be explored analytically for the genus 1 case, and it will be shown that non-singular solutions do indeed exist for open sets of the full parameter space. In section 6, the existence of regular solutions will be explored also for higher genus, in part by analytical and in part by numerical methods, and it will be shown for genus 2 that regular solutions exist in an open set of the full parameter space.

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4.11 Asymptotic behavior near the branch points

In this subsection, the asymptotic behavior of the above non-singular solutions near a branch point $u = e_i$ will be analyzed. Defining $u = e_i + z$, the harmonic functions h_1, h_2 behave as follows near $z = 0$,

$$\begin{aligned} h_1 &= 2i\left(\gamma_1 \frac{1}{\sqrt{z}} - \delta_1 \sqrt{z}\right) + o\left(z^{\frac{3}{2}}\right) + \text{c.c.} \\ h_2 &= 2\left(\gamma_2 \frac{1}{\sqrt{z}} - \delta_2 \sqrt{z}\right) + o\left(z^{\frac{3}{2}}\right) + \text{c.c.} \end{aligned} \tag{4.43}$$

where the constants are easily obtained from (4.8)

$$\begin{aligned} \gamma_1 &= \frac{P(e_i)Q_1(e_i)}{\prod_{j\neq i}(e_i - e_j)^{3/2}} \\ \gamma_2 &= \frac{P(e_i)Q_2(e_i)}{\prod_{j\neq i}(e_i - e_j)^{3/2}} \end{aligned} \tag{4.44}$$

and

$$\begin{aligned} \delta_1 &= \gamma_1 \left(\sum_k^p \left(\frac{1}{e_i - u_k} + \frac{1}{e_i - \bar{u}_k} \right) + \sum_k^q \frac{1}{q_i - \alpha_k} - \frac{1}{3} \sum_{k\neq i}^{2g-1} \frac{1}{e_k - e_i} \right) \\ \delta_2 &= \gamma_2 \left(\sum_k^p \left(\frac{1}{e_i - u_k} + \frac{1}{e_i - \bar{u}_k} \right) + \sum_k^q \frac{1}{q_i - \beta_k} - \frac{1}{3} \sum_{k\neq i}^{2g-1} \frac{1}{e_k - e_i} \right) \end{aligned} \tag{4.45}$$

Comparison with the genus 0 Janus solution (4.2) reveals that near each of the branch points the supergravity geometry approaches $AdS_5 \times S^5$ asymptotically. Introducing a new coordinate $z = e^{-2x-2iy}$ the asymptotic region near the branch point $z = 0$ is mapped to $x \to \infty$ and the coordinate $y \in [0, \pi/2]$.

4.11.1 The dilaton and the metric

The asymptotic values of the dilaton near each i-th branch point e_i is given by

$$e^{2\phi} = \left| \frac{Q_2(e_i)}{Q_1(e_i)} \right| + o(e^{-4x}) \tag{4.46}$$

which are constants which depend on e_i, α_b and β_b. The metric becomes

$$ds^2 = 4\sqrt{2\Delta}\Big(dx^2 + dy^2 + (\cos y)^2 ds^2_{S_1} + (\sin y)^2 ds^2_{S_2} + e^{2x} ds^2_{AdS_4}\Big) + o(e^{-2x}) \tag{4.47}$$

where

$$\Delta = \delta_2\gamma_1 - \delta_1\gamma_2 = \gamma_1\gamma_2 \left(\sum_{k=1}^q \frac{1}{e_i - \beta_k} - \frac{1}{e_i - \alpha_k} \right) \tag{4.48}$$

In the limit $x \to \infty$ the metric becomes $AdS_5 \times S^5$, and $4\sqrt{2\Delta}$ is the radius squared of the $AdS_5 \times S^5$ geometry in the neighborhood of e_i. Since there are $2g+2$ branch points (including infinity), the genus g solution has $2g + 2$ asymptotically $AdS_5 \times S^5$ regions, where the dilaton approaches (generically) different constant values. The holographic interpretation of this geometry will be presented in section 8.

4.11.2 The 2-form potential $B_{(2)}$

The functions b_1, b_2 (2.11) parametrize the NSNS and RR 2-form potentials of $B_{(2)}$. To compute b_1 and b_2, we need to evaluate the harmonic duals $\tilde{h}_1$ and $\tilde{h}_2$. Recall equation (9.24) of [1] for the harmonic functions h_1 and h_2, and equation (9.43) for their harmonic duals $\tilde{h}_1$ and $\tilde{h}_2$, in terms of the holomorphic functions $\mathcal{A}$ and $\mathcal{B}$,

$$\begin{aligned} h_1 &= -i(\mathcal{A} - \bar{\mathcal{A}}) & \tilde{h}_1 &= \mathcal{A} + \bar{\mathcal{A}} \\ h_2 &= \mathcal{B} + \bar{\mathcal{B}} & \tilde{h}_2 &= i(\mathcal{B} - \bar{\mathcal{B}}) \end{aligned} \tag{4.49}$$

The holomorphic functions are found as follows,

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_0 - 2\left(\gamma_1 \frac{1}{\sqrt{z}} - \delta_1 \sqrt{z}\right) + o(z^{\frac{3}{2}}) \\ \mathcal{B} &= i\mathcal{B}_0 + 2\left(\gamma_2 \frac{1}{\sqrt{z}} - \delta_2 \sqrt{z}\right) + o(z^{\frac{3}{2}}) \end{aligned} \tag{4.50}$$

Here, $\mathcal{A}_0$ and $\mathcal{B}_0$ are real z-independent parameters. They arise because the splitting of the harmonic functions h_1 and h_2 into holomorphic ones is unique only up to additive constants, which cannot be determined from the local properties of h_1 and h_2. (In the next subsection, the difference between the values of these constants at different branch points will be determined using Abelian integrals.) Putting all together, we have now the following asymptotic expressions for the fields b_1 and b_2,

$$\begin{aligned} b_1 &= -4\mathcal{B}_0 + \frac{32\Delta(\delta_2\gamma_1 + \delta_1\gamma_2)}{\gamma_1^2\gamma_2} (\sin y)^3 e^{-3x} + o(e^{-5x}) \\ b_2 &= -4\mathcal{A}_0 + \frac{32\Delta(\delta_2\gamma_1 + \delta_1\gamma_2)}{\gamma_2^2\gamma_1} (\cos y)^3 e^{-3x} + o(e^{-5x}) \end{aligned} \tag{4.51}$$

The associated 3-form fluxes vanish in the asymptotic $AdS_5 \times S^5$ regions.

4.12 Homology 3-spheres

The hyperelliptic solutions exhibit non-trivial 3-cycles on which the 3-form fields have non-zero charges. In this section, we determine these 3-cycles and evaluate the 3-form charges.

A non-trivial 3-cycle arises when a 1-parameter family of 2-spheres, either S_1^2 or S_2^2, starts at zero radius (respectively f_1 or f_2) and returns to zero radius in a manner consistent with the topology of 3-spheres (respectively S_1^3 or S_2^3). The relevant 1-parameter families correspond to intervals on the real line $\partial\Sigma$, located between consecutive branch points. The precise correspondence is as follows:

$$\begin{aligned} S_{1j}^3 &= \left\{[e_{2j}, e_{2j-1}] \times_f S_1^2\right\} & j &= 1, \cdots, g \\ S_{2j}^3 &= \left\{[e_{2j+1}, e_{2j}] \times_f S_2^2\right\} & j &= 1, \cdots, g \end{aligned} \tag{4.52}$$

Here, it is understood that the product $\times_f$ of the branch cut and a 2-sphere stands for a fibration of the 2-sphere over the interval, and not for a product of sets. Notice that for genus 0, which corresponds to the $AdS_5 \times S^5$ and Janus solutions, no non-trivial 3-cycles are found to exist.

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4.13 Evaluation of the RR and NSNS 3-form charges

The real NSNS and RR 3-form field strengths $H_{(3)}$ and $C_{(3)}$, are differentials of the complex 2-form potential, given by the relation $H_{(3)} + iC_{(3)} = dB_{(2)}$. The explicit expression for $B_{(2)}$ on our solutions is given by (2.10) and (2.11). As a result, the charges $\mathcal{H}_j$ and $\mathcal{C}_j$, respectively of the fields $H_{(3)}$ and $C_{(3)}$, across the non-trivial 3-cycles S^3_{1j} and S^3_{2j} are given as follows,

$$\begin{aligned} \mathcal{H}_j &\equiv \int_{S^3_{1j}} db_1 \wedge \hat{e}^{45} = +8\pi \int_{e_{2j}}^{e_{2j-1}} d\tilde{h}_2 \\ \mathcal{C}_j &\equiv \int_{S^3_{2j}} db_2 \wedge \hat{e}^{67} = -8\pi \int_{e_{2j+1}}^{e_{2j}} d\tilde{h}_1 \end{aligned} \tag{4.53}$$

In using the relation (2.11) between $b_{1,2}$ and $\tilde{h}_{1,2}$ in the above expressions, the second terms on the right hand sides of (2.11) cancel because $h_1 = 0$ on the intervals entering into the calculation of $\mathcal{H}_j$, while $h_2 = 0$ on the intervals entering the calculation of $\mathcal{C}_j$. An alternative way to see that the second terms on the right hand sides of (2.11) do not contribute is by observing that the above line integrals are effectively around closed curved on the full hyperelliptic Riemann surface (including the upper half plane and both Riemann sheets), and that those contributions to b_1 and b_2 are single-valued, and thus cancel out of the integrals.

To evaluate the line integrals over $d\tilde{h}_{1,2}$, we use (2.12) to express the differentials of $\tilde{h}_{1,2}$ in terms of those of $h_{1,2}$, and then use (4.31) to evaluate the line integrals. The functions $q_1(u)$ and $q_2(u)$ of (4.24), which enter in the expression (4.31), are single-valued scalars and do not contribute to the line integrals of (4.53). The remaining integrals give,

$$\begin{aligned} \mathcal{H}_j &= -16i\pi \int_{e_{2j}}^{e_{2j-1}} \frac{p_2(u)\,du}{s(u)} \\ \mathcal{C}_j &= -16\pi \int_{e_{2j+1}}^{e_{2j}} \frac{p_1(u)\,du}{s(u)} \end{aligned} \tag{4.54}$$

Both integrals are real, given the phase of $s(u)$ in (4.27), and generically non-vanishing.

5. Genus 1 solutions

The elliptic case provides the simplest solution of the hyperelliptic Ansatz that goes beyond the $AdS_5 \times S^5$ and Janus solutions. Generically, it will have four distinct asymptotic $AdS_5 \times S^5$ regions, each with a different value of the dilaton. The Abelian integrals may be recast in terms of the familiar elliptic functions on the torus,[6] and the domain of parameter space that leads to non-singular solutions may be constructed explicitly and analytically.

[6]Useful general references on elliptic functions, conformal mapping, and explicit formulas may be found in the Bateman manuscript [12], and in [13].

5.1 Formulation on the lower half-plane

First, the parametrization given in the preceding section for all genera simplifies considerably at genus 1, and reduces to the following,

$$\begin{aligned} s(u)^2 &= (u-e_1)(u-e_2)(u-e_3) \\ P(u) &= (u-u_1)(u-\bar{u}_1) \\ Q_1(u) &= (u-\alpha_1)(u-\alpha_2) \\ Q_2(u) &= (u-\beta_1)(u-\beta_2) \end{aligned} \tag{5.1}$$

where we may choose $e_1+e_2+e_3=0$, without loss of generality. The branch points e_1, e_2, e_3 and the roots $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real, and subject to the ordering relation (4.40) for $g=1$,

$$\alpha_2 < e_3 < \beta_2 < e_2 < \alpha_1 < e_1 < \beta_1 \tag{5.2}$$

and $\mathrm{Im}(u_1)<0$. We use the calculations of subsection 4.5 to derive the period relations in the elliptic case. To this end, we compute $G_i(u) = -u+e_i$, as well as the following objects,

$$\begin{aligned} p_1(u) &= u - A_4 - \sum_{i=1}^{3} A_i(u-e_i) \\ p_2(u) &= u - B_4 - \sum_{i=1}^{3} B_i(u-e_i) \end{aligned} \tag{5.3}$$

The constants A_i, B_i are obtained as residues and are given by

$$\begin{aligned} A_i &= -P(e_i)Q_1(e_i)\,(E_i)^{-2} \\ B_i &= -P(e_i)Q_2(e_i)\,(E_i)^{-2} \end{aligned} \tag{5.4}$$

while the constants A_4 and B_4 may be obtained from the next-to-leading order behavior at $u=\infty$ and are found to be,

$$\begin{aligned} A_4 &= \alpha_1+\alpha_2+u_1+\bar{u}_1 \\ B_4 &= \beta_1+\beta_2+u_1+\bar{u}_1 \end{aligned} \tag{5.5}$$

Here, E_i is given solely in terms of the branch points by $E_i \equiv (e_i-e_j)(e_i-e_k)$ where e_j and e_k are two distinct branch points which are distinct also from e_i. The period relations of (4.33) take the following form,

$$\begin{aligned} \mathrm{Im}\left(\int_{e_2}^{e_1} \frac{p_1(u)du}{s(u)}\right) &= 0 \\ \mathrm{Re}\left(\int_{e_3}^{e_2} \frac{p_2(u)du}{s(u)}\right) &= 0 \end{aligned} \tag{5.6}$$

They may be rendered more explicit by using the expressions for p_1 and p_2 of (5.3), as well as the following basic elliptic integrals,

$$\int_{e_2}^{e_1} \frac{du}{s(u)} = \omega_3 \qquad\qquad \int_{e_2}^{e_1} \frac{u\,du}{s(u)} = -\zeta(\omega_3)$$
$$\int_{e_3}^{e_2} \frac{du}{s(u)} = \omega_1 \qquad\qquad \int_{e_3}^{e_2} \frac{u\,du}{s(u)} = -\zeta(\omega_1) \tag{5.7}$$

where $\zeta(u)$ is the Weierstrass ζ-function. In view of (4.27), the periods ω_1 and $\zeta(\omega_1)$ are real, while the periods ω_3 and $\zeta(\omega_3)$ are purely imaginary. The period relations may then be recast as follows,

$$A_4 = -\zeta_3 + \sum_{i=1}^{3} A_i\,(e_i + \zeta_3) \qquad\qquad \zeta_3 \equiv \frac{\zeta(\omega_3)}{\omega_3}$$
$$B_4 = -\zeta_1 + \sum_{i=1}^{3} B_i\,(e_i + \zeta_1) \qquad\qquad \zeta_1 \equiv \frac{\zeta(\omega_1)}{\omega_1} \tag{5.8}$$

Together with the defining relations for A_i, B_i, A_4 and B_4, the period relations give two real equations which may be viewed as equations for the complex zero u_1, as a function of the real modulus of the torus (parametrized by one of the branch points e_1, e_2, e_3) and the real zeros $\alpha_1, \alpha_2, \beta_1, \beta_2$. The relations implied on u_1 are quadratic and of the following form,

$$a_0|u_1|^2 - a_1(u_1 + \bar{u}_1) + a_2 = 0 \qquad\qquad \mathrm{Im}(u_1) < 0$$
$$b_0|u_1|^2 - b_1(u_1 + \bar{u}_1) + b_2 = 0 \tag{5.9}$$

where the coefficients of these quadrics are given by

$$a_n = -\delta_{n,1} + (\alpha_1 + \alpha_2 + \zeta_3)\delta_{n,2} + \sum_{i=1}^{3} e_i^n (e_i + \zeta_3) Q_1(e_i) E_i^{-2}$$
$$b_n = -\delta_{n,1} + (\beta_1 + \beta_2 + \zeta_1)\delta_{n,2} + \sum_{i=1}^{3} e_i^n (e_i + \zeta_1) Q_2(e_i) E_i^{-2} \tag{5.10}$$

for $n = 0, 1, 2$ and $\delta_{n,1}$ and $\delta_{n,2}$ are the Kronecker δ. The quadrics of (5.9) are two half-circles whose centers lie on the real axis.

5.2 Parameter space via four real zeros

To investigate the existence of solutions to (5.9) and (5.10), it is convenient to recast (5.9) in terms of the centers and radii of the circles,

$$\left| u_1 - \frac{a_1}{a_0} \right|^2 = R_a^2 \qquad\qquad R_a^2 = \frac{a_1^2 - a_0 a_2}{a_0^2}$$
$$\left| u_1 - \frac{b_1}{b_0} \right|^2 = R_b^2 \qquad\qquad R_b^2 = \frac{b_1^2 - b_0 b_2}{b_0^2} \tag{5.11}$$

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The necessary and sufficient conditions for the existence of a solution are that these circles have positive R_a^2 and R_b^2, and have non-trivial intersection. When they do, the intersection will produce a single point u_1, which is the unique common complex zero of the differentials ∂h_1 and ∂h_2 in the lower half-plane. In summary, these conditions are

$$0 \leq R_a^2, R_b^2 \qquad\qquad \left|\frac{a_1}{a_0} - \frac{b_1}{b_0}\right| \leq R_a + R_b \tag{5.12}$$

In the next subsection, we shall show that the coefficients a_n, b_n may be expressed in terms of genus 1 modular forms.

5.3 Parameter space via the complex and two real zeros

A more practical approach is to use the complex zero itself as a parameter, together with the two real zeros α_1, β_2 whose range is compact. We may then solve for α_2 and β_1, using the relations (5.9) together with (5.10). One obtains,

$$\begin{aligned} \alpha_2 &= \frac{\alpha_1 + u_1 + \bar{u}_1 + \zeta_3 + \sum_{i=1}^3 (e_i + \zeta_3) e_i (e_i - \alpha_1) |u_1 - e_i|^2 E_i^{-2}}{-1 + \sum_{i=1}^3 (e_i + \zeta_3)(e_i - \alpha_1)|u_1 - e_i|^2 E_i^{-2}} \\ \beta_1 &= \frac{\beta_2 + u_1 + \bar{u}_1 + \zeta_1 + \sum_{i=1}^3 (e_i + \zeta_1) e_i (e_i - \beta_2) |u_1 - e_i|^2 E_i^{-2}}{-1 + \sum_{i=1}^3 (e_i + \zeta_1)(e_i - \beta_2)|u_1 - e_i|^2 E_i^{-2}} \end{aligned} \tag{5.13}$$

subject to the range (5.2). Considering the modulus as given, then for each set of values α_1 and β_2 in the range (5.2), the above expression, together with the ranges for α_2 and β_1, limit the domain of u_1 in the complex plane by two semi-circles. The allowed range of the parameters will be obtained analytically for the case of the square torus, where many simplifications occur.

5.4 Formulation in terms of elliptic functions and explicit solution

An entire genus 1 Riemann surface, i.e. the double cover of the plane, is uniformized by the Weierstrass function which maps the torus[7] with half-periods ω_1 and ω_3, and modulus $\tau = \omega_3/\omega_1$ into the double cover of the plane by the map

$$z \longrightarrow (\wp(z), \wp'(z)) \tag{5.14}$$

where $\wp'(z)$ is given in terms of $\wp(z)$ through the defining equation,

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) \tag{5.15}$$

up to a sign. This sign distinguishes between the upper and lower Riemann sheets. The branch points are related to the half-periods by $e_i = \wp(\omega_i)$, for $i = 1, 2, 3$, and $\infty = \wp(0)$.

The Weierstrass function $\wp(z)$ is periodic with periods $2\omega_1$ and $2\omega_3$. It is real if and only if either $z = m\omega_1 + i\rho$, or $z = n\omega_3 + \rho$, both with ρ real, and m and n arbitrary integers. This divides the fundamental region for the torus into 4 regions (see figure 4). Under $z \rightarrow 2\omega_1 + 2\omega_3 - z$, the torus is mapped into itself, $\wp \rightarrow \wp$ but $\wp' \rightarrow -\wp'$, so

[7] We adopt the conventions of [12] with $\omega = \omega_1$ real, and $\omega' = \omega_3$, and $\tau = \omega_3/\omega_1$ purely imaginary.

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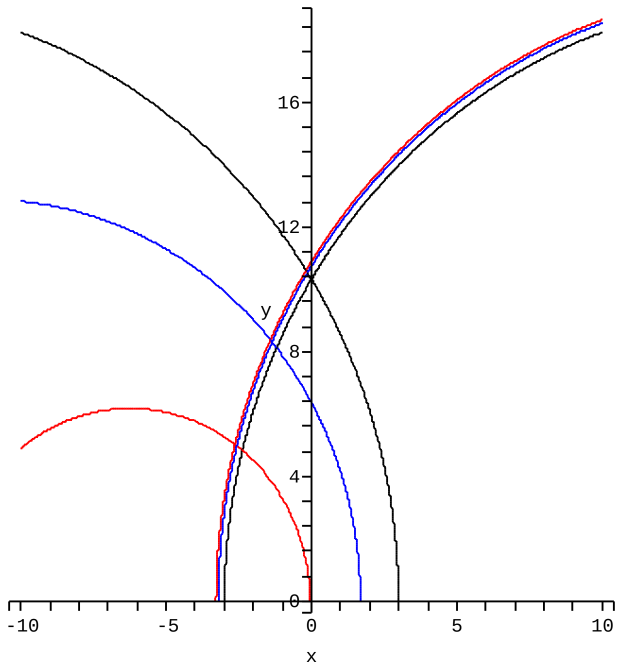

Figure 3: Allowed parameter region for the complex zero $u_1 = x + iy$ for given values of α_1 and β_2 (equal to $(e_1 + e_2)/2$ and $(e_2 + e_3)/2$ in this case), and varying $\tau = i$ (black arcs), $1.2i$ (blue arcs), and $1.6i$ (red arcs). The allowed regions are the trigons bounded by the real axis and arcs of the same color.

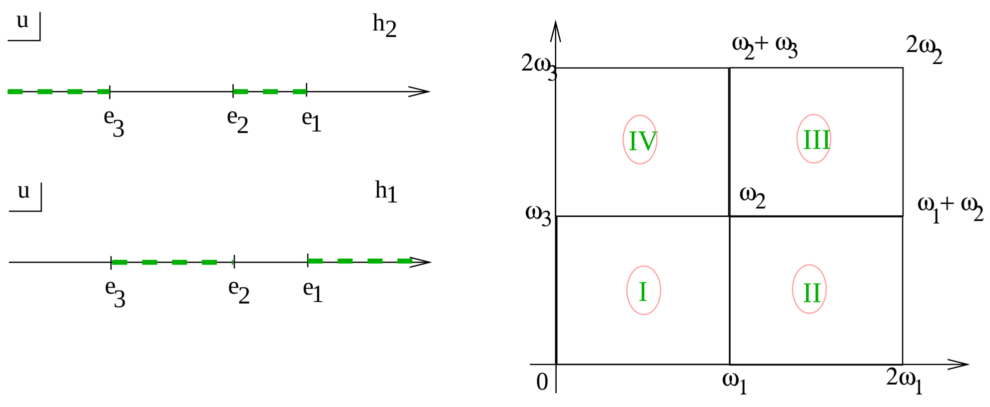

Figure 4: The cut plane and fundamental domain of the elliptic solution.

that the two Riemann sheets are interchanged, and regions I and III, as well as II and IV, are interchanged with one another... Furthermore, regions I and III map to the lower half-plane, while regions II and IV map to the upper half-plane. The lower half-plane of a single sheet is the image of region I. Using the map $u = \wp(z)$, the inversion formulas [12],

$$\wp(z+\omega_i) = e_i + \frac{E_i}{\wp(z) - e_i} \qquad\qquad i = 1,2,3 \tag{5.16}$$

and the sign choice $s(u) = \wp'(z)/2$, the differentials ∂h_1 and ∂h_2 may be recast in terms of the Weierstrass function,

$$\begin{aligned}
\partial h_1 &= i \sum_{\alpha=0}^{3} A_\alpha \Big(\wp(z+\omega_\alpha) + \zeta_3 \Big) dz \\
\partial h_2 &= \sum_{\beta=0}^{3} B_\beta \Big(\wp(z+\omega_\beta) + \zeta_1 \Big) dz
\end{aligned} \tag{5.17}$$

where $A_0 = B_0 = -1$. With the help of the relation between the Weierstrass functions, $\zeta'(z) = -\wp(z)$, it is immediate to derive the expressions of the harmonic functions h_1 and h_2 themselves. The overall additive integration constant generated in the process is fixed by the requirement of vanishing Dirichlet boundary conditions, on the segments $[0,\omega_1]$ and $[\omega_3,\omega_2]$ for h_1 and on the segments $[0,\omega_3]$ and $[\omega_1,\omega_2]$ for h_2, and we find,

$$\begin{aligned}
h_1(z) &= 2i\zeta(\omega_3)(A_2 + A_3) - \sum_{\alpha=0}^{3} iA_\alpha \left[\zeta(z+\omega_\alpha) - \zeta(\bar z + \bar\omega_\alpha) - (z - \bar z)\zeta_3\right] \\
h_2(z) &= 2\zeta(\omega_1)(B_1 + B_2) - \sum_{\alpha=0}^{3} B_\alpha \left[\zeta(z+\omega_\alpha) + \zeta(\bar z + \bar\omega_\alpha) - (z + \bar z)\zeta_1\right]
\end{aligned} \tag{5.18}$$

To show the Dirichlet vanishing on the segments $[\omega_3,\omega_2]$ and $[\omega_1,\omega_2]$, we have made use of the following addition formula for the ζ-function [12],

$$\zeta(z+z') = \zeta(z) + \zeta(z') + \frac{1}{2}\frac{\wp'(z) - \wp'(z')}{\wp(z) - \wp(z')} \tag{5.19}$$

and the fact that $\wp'(\omega_i) = 0$ for $i = 1,2,3$.

5.5 Regularity

The general arguments of the preceding section guarantee that $W < 0$ as long as the ordering (5.2) holds, and that h_1 and h_2 are positive on their respective Neumann segments. Positivity of h_1 and h_2 was argued on general grounds as well, but in the elliptic case, it may actually be given a solid proof using explicit formulas. The starting point is two expansion formulas for the Weierstrass ζ-function,

$$\begin{aligned}
\zeta(z) &= z\zeta_1 + \frac{\pi}{2\omega_1} \sum_{m=-\infty}^{+\infty} \text{cotg}\left(\frac{\pi z}{2\omega_1} + m\pi\frac{\omega_3}{\omega_1}\right) \\
\zeta(z) &= z\zeta_3 + \frac{\pi}{2\omega_3} \sum_{m=-\infty}^{+\infty} \text{cotg}\left(\frac{\pi z}{2\omega_3} + m\pi\frac{\omega_1}{\omega_3}\right)
\end{aligned} \tag{5.20}$$

The first of these formulas is familiar [13], while the second may be obtained from the first by interchanging the half-periods ω_1 and ω_3. The first formula will be applied to h_2,

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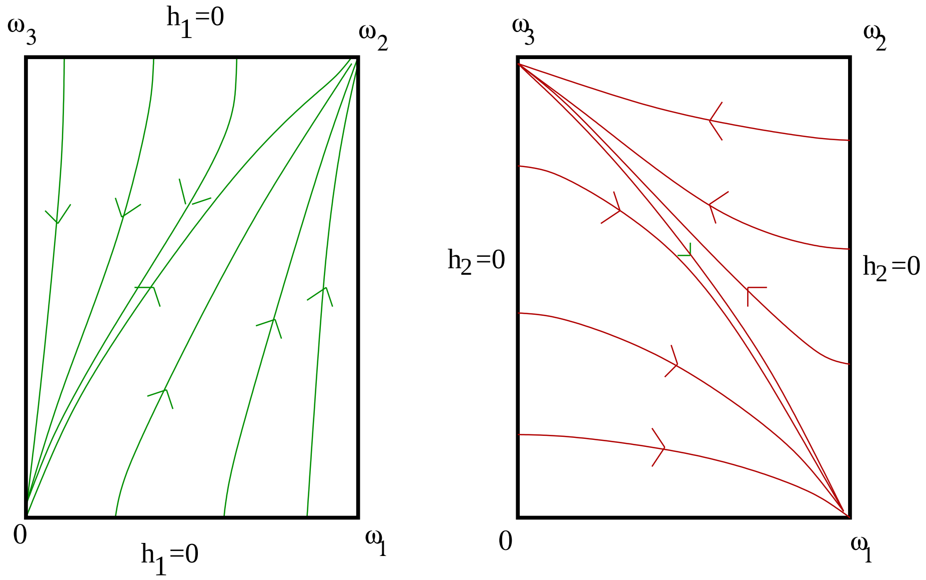

Figure 5: Field lines on the square of half-periods.

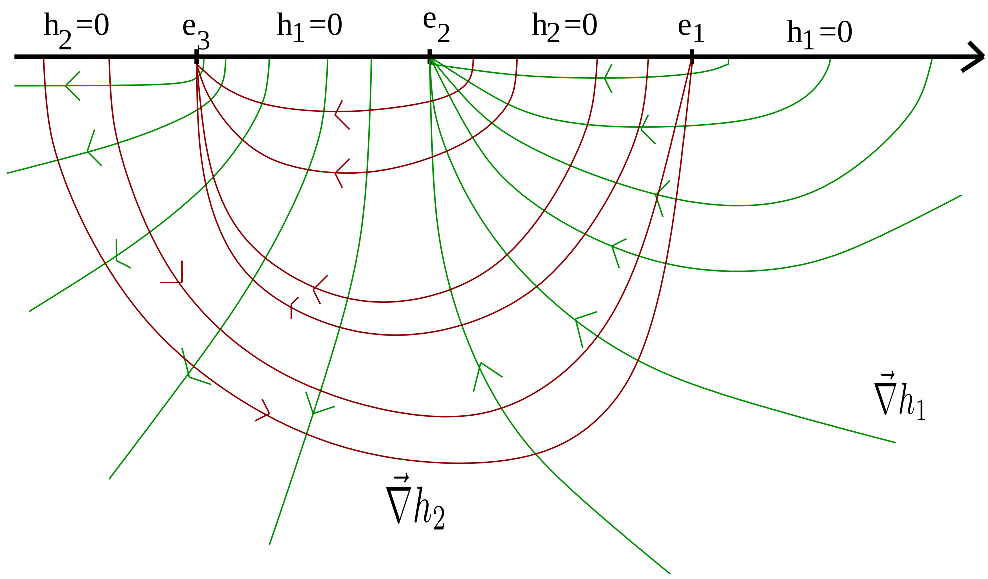

Figure 6: Field lines on the half-plane.

the second to h_1. Remarkably, all dependence on $\zeta(\omega_1)$ and $\zeta(\omega_3)$ cancels, and h_1 and h_2 are expressed as simple infinite series expansions. Using the formula $\operatorname{cotg} u + \operatorname{cotg} \bar{u} = \sin(u+\bar{u})/|\sin u|^2$, complex conjugate pairs of terms may be combined, to obtain the following formulas,

$$
\begin{aligned}
h_1(z) &= -\frac{\pi i}{2\omega_3}\sum_{\alpha=0}^{3} A_\alpha \sum_{m=-\infty}^{+\infty} \frac{\sin\left(\frac{\pi}{2\omega_3}(z-\bar z+\omega_\alpha-\bar\omega_\alpha)\right)}{|\sin\left(\frac{\pi}{2\omega_3}(z+\omega_\alpha+2m\omega_1)\right)|^2} \\
h_2(z) &= -\frac{\pi}{2\omega_1}\sum_{\alpha=0}^{3} B_\alpha \sum_{m=-\infty}^{+\infty} \frac{\sin\left(\frac{\pi}{2\omega_1}(z+\bar z+\omega_\alpha+\bar\omega_\alpha)\right)}{|\sin\left(\frac{\pi}{2\omega_1}(z+\omega_\alpha+2m\omega_3)\right)|^2}
\end{aligned} \tag{5.21}
$$

Notice that the sin factors in the numerators are actually independent of the summation index m and may be moved out of the sum over m. Since these numerators depend only on either $z-\bar z$ for h_1 and $z+\bar z$ for h_2, it is immediate that positivity on their Neumann boundary segments actually implies positivity everywhere in the fundamental region bounded by the half-periods, which thus proves that $h_1, h_2 > 0$ everywhere.

5.6 Analytical solution and modular polynomials

The solution (5.18) is completely explicit in terms of the branch points e_i, and the real zeros $\alpha_1, \alpha_2, \beta_1, \beta_2$, chosen subject to (5.2), as long as the period relations (5.8) are solved, by satisfying (5.9) with (5.10). The elliptic parametrization will allow us to render the period relations more explicit by expressing the parameters a_n and b_n of (5.10) in terms of modular forms. Defining the modular objects,

$$
\begin{aligned}
M_1^{(m)} &= \sum_{i=1}^{3} e_i^m (e_i+\zeta_3)E_i^{-2} \\
M_2^{(m)} &= \sum_{i=1}^{3} e_i^m (e_i+\zeta_1)E_i^{-2}
\end{aligned} \tag{5.22}
$$

the parameters a_n and b_n of (5.10) may be expressed as follows,

$$
\begin{aligned}
a_n &= \alpha_1\alpha_2 M_1^{(n)} + (\alpha_1+\alpha_2)\left(\delta_{n,2} - M_1^{(n+1)}\right) - \delta_{n,1} + \zeta_3\delta_{n,2} + M_1^{(n+2)} \\
b_n &= \beta_1\beta_2 M_2^{(n)} + (\beta_1+\beta_2)\left(\delta_{n,2} - M_2^{(n+1)}\right) - \delta_{n,1} + \zeta_1\delta_{n,2} + M_2^{(n+2)}
\end{aligned} \tag{5.23}
$$

To compute the modular objects M_1 and M_2, we express the branch points in terms of genus one ϑ-functions, using the Thomae formulas [12],

$$
\begin{aligned}
e_1 - e_2 &= \frac{\pi^2}{4\omega_1^2}\vartheta_4^4 = -\frac{i\pi}{\omega_1^2}\partial_\tau \ln\frac{\vartheta_2}{\vartheta_3} \\
e_1 - e_3 &= \frac{\pi^2}{4\omega_1^2}\vartheta_3^4 = -\frac{i\pi}{\omega_1^2}\partial_\tau \ln\frac{\vartheta_2}{\vartheta_4} \\
e_2 - e_3 &= \frac{\pi^2}{4\omega_1^2}\vartheta_2^4 = -\frac{i\pi}{\omega_1^2}\partial_\tau \ln\frac{\vartheta_3}{\vartheta_4}
\end{aligned} \tag{5.24}
$$

The relation $e_1+e_2+e_3=0$ holds in view of the famous Jacobi identity $\vartheta_2^4+\vartheta_4^4=\vartheta_3^4$. The branch points are modular forms under the subgroup of the full modular group

SL(2, **Z**) which leaves the half-periods invariant. With these results, we readily compute the combinations E_i,

$$E_1^2(e_2-e_3)^2 = E_2^2(e_3-e_1)^2 = E_3^2(e_1-e_2)^2 = 2^8\pi^{12}\eta(\tau)^{24} \tag{5.25}$$

where $2\eta^3 = \vartheta_2\vartheta_3\vartheta_4$ is the Dedekind η-function. It remains to compute the combinations $\zeta_{1,3}$ defined in (5.8). The starting point is [13],

$$\zeta_1 = \zeta_3 + \frac{i\,\pi}{2\omega_1\omega_3} = -\frac{i\pi}{\omega_1^2}\partial_\tau \ln\eta(\tau) \tag{5.26}$$

which shows that ζ_1 and its $\tau \to -1/\tau$ transform ζ_3 are modular connections. Using the relations between ϑ^4 and $\partial_\tau \ln\vartheta$ in the second column of (5.24), we also obtain the following formulas for $i = 1, 2, 3$,

$$e_i + \zeta_1 = -\frac{i\pi}{\omega_1^2}\partial_\tau \ln\vartheta_{i+1}(0,\tau) \tag{5.27}$$

We conclude by noticing that the modular objects $M_{1,2}^{(m)}$ are not modular forms, but modular connections, as the parts proportional to ζ_1 and ζ_3 transform inhomogeneously under the modular group SL(2, **Z**).

5.7 The special case of the square torus

For the square torus, $\tau = i$, the above quantities may be evaluated in elementary terms and the period relations may be solved explicitly. Without loss of generality, we choose the half-periods in the canonical normalization, $\omega_1 = 1/2$ and $\omega_3 = \tau/2$. Using the symmetry of the square torus, we have $\zeta_3 = -\zeta_1$. Using the first equation in (5.26), we find $\zeta_1 = \pi$ and $\zeta_3 = -\pi$. In terms of the parameters of the lower half-plane representation, we have

$$\begin{aligned} e_1 &= -e_3 = k > 0 & \qquad e_2 &= 0 \\ E_1 &= E_3 = 2k^2 & \qquad E_2 &= -k^2 \end{aligned} \tag{5.28}$$

where $k \equiv \wp(1/2)$, whose numerical value is approximately $k = 6.875185816$. The ranges of the real zeros are as follows,

$$\alpha_2 < -k < \beta_2 < 0 < \alpha_1 < k < \beta_1 \tag{5.29}$$

The modular objects $M_1^{(m)}$ and $M_2^{(m)}$, defined in (5.22) take the form,

$$\begin{aligned} 2M_1^{(2m)} &= -\pi k^{2m-4}\,(1+2\delta_{m,0}) \\ 2M_1^{(2m+1)} &= k^{2m-2} \\ 2M_2^{(2m)} &= +\pi k^{2m-4}\,(1+2\delta_{m,0}) \\ 2M_2^{(2m+1)} &= k^{2m-2} \end{aligned} \tag{5.30}$$

and are used to calculate the parameters a_n and b_n in the period relations (5.9), and we find,

$$\begin{aligned}
2k^4 a_0 &= -3\pi\alpha_1\alpha_2 - (\alpha_1+\alpha_2)k^2 - \pi k^2 \\
2k^4 b_0 &= +3\pi\beta_1\beta_2 - (\beta_1+\beta_2)k^2 + \pi k^2 \\
2k^2 a_1 &= +\alpha_1\alpha_2 + \pi(\alpha_1+\alpha_2) - k^2 \\
2k^2 b_1 &= +\beta_1\beta_2 - \pi(\beta_1+\beta_2) - k^2 \\
2k^2 a_2 &= -\pi\alpha_1\alpha_2 + (\alpha_1+\alpha_2)k^2 - 3\pi k^2 \\
2k^2 b_2 &= +\pi\beta_1\beta_2 + (\beta_1+\beta_2)k^2 + 3\pi k^2
\end{aligned} \tag{5.31}$$

One could now proceed and impose the conditions (5.12) for the existence of solutions.

It turns out that a more explicit description of the allowed parameter space may be obtained by leaving the complex zero u_1 as a known parameter and solving instead for the real zeros α_2 and β_1, as was done also in subsection 5.3. It is convenient to scale a factor of k out of u_1,

$$u_1 = kv \tag{5.32}$$

In terms of v in the lower half-plane, $\mathrm{Im}(v) < 0$, the relations (5.9) are linear in α_2 and β_1, and are solved as follows,

$$\begin{aligned}
\alpha_2 &= -\frac{\pi k^2(|v|^2+3) + \alpha_1 k^2(|v|^2-1) + (\pi\alpha_1 - k^2)k(v+\bar v)}{\pi\alpha_1(3|v|^2+1) + k^2(|v|^2-1) + (\alpha_1+\pi)k(v+\bar v)} \\
\beta_1 &= \frac{\pi k^2(|v|^2+3) - \beta_2 k^2(|v|^2-1) + (\pi\beta_2 + k^2)k(v+\bar v)}{-\pi\beta_2(3|v|^2+1) + k^2(|v|^2-1) - (\pi-\beta_2)k(v+\bar v)}
\end{aligned} \tag{5.33}$$

These results must be supplemented with the inequalities

$$\begin{aligned}
\alpha_2 &< -k \\
\beta_1 &> +k
\end{aligned} \tag{5.34}$$

For the special point $v = -i$, it is clear that both inequalities (5.34) hold for all allowed values $-k < \beta_2 < 0 < \alpha_1 < k$. In the next subsection, we shall show that the corresponding solution may actually be mapped onto the Janus solution.

For non-special points $v \neq -i$, the solution is distinct from Janus. Given the ranges $-k < \beta_2 < 0 < \alpha_1 < k$ with strict inequalities, it is clear from (5.34) that there will exist an open set containing $v = -i$ for which the inequalities will be satisfied, and thus regular solutions will exist.

5.8 Supersymmetric Janus as a limiting case

We now return to the general torus in the formulation of (5.17) and (5.18). There is one type of symmetric assignment of the A_α and B_α for which all inequalities and periods relations are automatically satisfied, and which precisely reproduces the genus 0 Janus solution. The assignment is given as follows,

$$\begin{aligned}
A_0 &= -A_2 = -1 & \qquad B_0 &= -B_2 = -1 \\
A_1 &= -A_3 = -a & \qquad B_1 &= -B_3 = +b
\end{aligned} \tag{5.35}$$

where $a, b > 0$. In ∂h_1 and ∂h_2, the dependences on ζ cancel, and one is left with

$$\begin{aligned}\partial h_1(z) &= -i\wp(z) + i\wp(z+\omega_2) - ia\wp(z+\omega_1) + ia\wp(z+\omega_3) \\ \partial h_2(z) &= -\wp(z) + \wp(z+\omega_2) + b\wp(z+\omega_1) - b\wp(z+\omega_3)\end{aligned} \tag{5.36}$$

Since $\partial h_1(\omega_2/2) = \partial h_2(\omega_2/2) = 0$, the point $z = \omega_2/2$ is the common complex zero $u_1 = \wp(\omega_2/2)$. (For the square torus, and using formula (15) of section 13.13 of [12], we find indeed that $u_1 = -ik$.) To identify this solution with Janus, we compute the functions h_1 and h_2 for this assignment, with the help of (5.18). Using the formula (5.19), as well as the fact that $\zeta(\omega_1)$ is real and $\zeta(\omega_3)$ is imaginary, we recast the functions as follows,

$$\begin{aligned}h_1(z) &= \frac{i}{2}\wp'(z)\left(-\frac{1}{\wp(z)-e_2} + \frac{a}{\wp(z)-e_1} - \frac{a}{\wp(z)-e_3}\right) + \text{c.c.} \\ h_2(z) &= \frac{1}{2}\wp'(z)\left(-\frac{1}{\wp(z)-e_2} - \frac{b}{\wp(z)-e_1} + \frac{b}{\wp(z)-e_3}\right) + \text{c.c.}\end{aligned} \tag{5.37}$$

We now identify with the variables of the Janus solution u, r, by identifying the functions h_1 and h_2, which requires the following map,

$$\begin{aligned}\sqrt{u} &= -\frac{1}{2}\frac{\wp'}{\wp - e_2} \\ \frac{1}{\sqrt{u}} &= -\frac{b}{2}\frac{\wp'}{\wp - e_1} + \frac{b}{2}\frac{\wp'}{\wp - e_3} \\ -\frac{r}{\sqrt{u}} &= +\frac{a}{2}\frac{\wp'}{\wp - e_1} - \frac{a}{2}\frac{\wp'}{\wp - e_3}\end{aligned} \tag{5.38}$$

Consistency of these three equations requires first the proportionality of the last two equations, or, $r = a/b$, as well as the relation obtained by taking the sidewise product of the first two equations. After simplification of the $\wp$-dependence, using (5.15), we have $1 = b(e_1 - e_3)$. This is always possible since, by our conventions, we have $e_1 > e_3$.

5.9 Homology 3-spheres and 3-form charges

The homology 3-spheres in this geometry are $S^3_1 = [e_2, e_1] \times_f S^2_1$ and $S^3_2 = [e_3, e_2] \times_f S^2_2$. The corresponding charges are

$$\begin{aligned}\mathcal{H} &= \frac{8\pi^2}{\omega_1}\left(1 - \sum_{i=1}^{3} B_i\right) \\ \mathcal{C} &= \frac{8\,i\,\pi^2}{\omega_3}\left(1 - \sum_{i=1}^{3} A_i\right)\end{aligned} \tag{5.39}$$

Notice that these vansih on the Janus solution, but are otherwise generally non-vanishing.

6. Higher genus solutions

At genus higher than 1, the explicit solution involves hyperelliptic integrals, whose explicit form is less familiar than that of elliptic functions. Nonetheless, partial analytic and

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numerical study of the solutions is possible. In subsection 6.1, the general set-up needed to obtain regular solutions at genus 2 is presented, while in subsection 6.4 a general continuity argument is given for the existence of completely regular solutions, which satsify all the positivity conditions, at all genera. Finally, the solution may be formulated in terms of higher genus ϑ-functions, which is done in subsection 6.3.

6.1 The genus 2 solutions

We follow closely the construction given in subsections 4.5 and 4.7 valid for any genus. We normalize the genus 2 curve so that

$$\begin{aligned} s(u)^2 &= (u-e_1)(u-e_2)(u-e_3)(u-e_4)(u-e_5) \\ 0 &= e_1+e_2+e_3+e_4+e_5 \\ 2g_2 &= e_1^2+e_2^2+e_3^2+e_4^2+e_5^2 \end{aligned} \tag{6.1}$$

It is straightforward to compute $G_i(u)$, and we find,

$$G_i(u) = (u-e_i)\left(-3u^2-4ue_i-3e_i^2+g_2\right) \tag{6.2}$$

The complex zeros $u_1, u_2, \bar u_1, \bar u_2$, with $\mathrm{Im}(u_1), \mathrm{Im}(u_2) < 0$, and the real zeros $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$, enter as follows,

$$\begin{aligned} P(u) &= (u-u_1)(u-\bar u_1)(u-u_2)(u-\bar u_2) \\ Q_1(u) &= (u-\alpha_1)(u-\alpha_2)(u-\alpha_3) \\ Q_2(u) &= (u-\beta_1)(u-\beta_2)(u-\beta_3) \end{aligned} \tag{6.3}$$

Positivity of h_1, h_2 and negativity of W require the following ordering,

$$\alpha_3 < e_5 < \beta_3 < e_4 < \alpha_2 < e_3 < \beta_2 < e_2 < \alpha_1 < e_1 < \beta_1 \tag{6.4}$$

The polynomials $R_1(u)$ and $R_2(u)$ are obtained from the expansions at $u=\infty$, and we find,

$$\begin{aligned} R_1(u) &= u^2 - A_7 u + A_6 \\ R_2(u) &= u^2 - B_7 u + B_6 \end{aligned} \tag{6.5}$$

where the coefficients are given by

$$\begin{aligned} A_7 &= 2\mathrm{Re}(u_1+u_2) + \alpha_1+\alpha_2+\alpha_3 \\ B_7 &= 2\mathrm{Re}(u_1+u_2) + \beta_1+\beta_2+\beta_3 \\ A_6 &= |u_1|^2+|u_2|^2+4\mathrm{Re}(u_1)\,\mathrm{Re}(u_2)+2\mathrm{Re}(u_1+u_2)(\alpha_1+\alpha_2+\alpha_3) \\ &\quad +g_2+\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 \\ B_6 &= |u_1|^2+|u_2|^2+4\mathrm{Re}(u_1)\,\mathrm{Re}(u_2)+2\mathrm{Re}(u_1+u_2)(\beta_1+\beta_2+\beta_3) \\ &\quad +g_2+\beta_1\beta_2+\beta_2\beta_3+\beta_3\beta_1 \end{aligned} \tag{6.6}$$

Putting all together, the period relations are as follows,

$$k = 1, 2 \qquad \int_{e_{2k}}^{e_{2k-1}} \frac{du}{s(u)} \left[u^2 - A_7 u + A_6 + \sum_{i=1}^{5} A_i G_i(u) \right] = 0$$

$$k = 1, 2 \qquad \int_{e_{2k+1}}^{e_{2k}} \frac{du}{s(u)} \left[u^2 - B_7 u + B_6 + \sum_{i=1}^{5} B_i G_i(u) \right] = 0 \qquad (6.7)$$

where

$$\begin{aligned} A_i &= -P(e_i) Q_1(e_i) \, (E_i)^{-2} \\ B_i &= -P(e_i) Q_2(e_i) \, (E_i)^{-2} \end{aligned} \qquad (6.8)$$

The four period integrals above take on either real or purely imaginary values, as a result of the phase values of (4.27) for $s(u)$. Since the $s(u)$ retains constant phase inside any interval between consecutive branch points, the denominator $s(u)$ in the period integrals above may be replaced by $|s(u)|$. Given the moduli of the surface, and the real zeros $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$, these 4 real equations then determine the two complex zeros u_1 and u_2.

6.2 Numerical analysis

We have explored the solutions of these equations numerically for genus 2 surfaces with various degrees of $\mathbf{Z}_2$ symmetry, obtained via the following arrangement of the branch points,

$$e_5 = -t_2, \qquad e_4 = -\frac{1}{t_2}, \qquad e_3 = 0, \qquad e_2 = \frac{1}{t_1}, \qquad e_1 = t_1 \qquad (6.9)$$

for $t_1, t_2 > 1$. The first $\mathbf{Z}_2$ is the hyperelliptic involution, while a second $\mathbf{Z}_2$ is the inversion $u \to 1/u$. An extra $\mathbf{Z}_2$, given by $u \to -u$, is obtained by setting $t_2 = t_1$, while a Riemann surface with symmetry $(\mathbf{Z}_2)^4$ is obtained when $t_1 = t_2 = t_0$, where $t_0 \equiv 1.27201965$. The period matrix for the surface with $(\mathbf{Z}_2)^4$ symmetry assumes the symmetrical form,

$$\Omega_{IJ} = \begin{pmatrix} 0.89442719\, i & -0.44721359\, i \\ 0.44721359\, i & 0.89442719\, i \end{pmatrix} \qquad I, J = 1, 2 \qquad (6.10)$$

In figure 7, we make a choice for the real zeros indicated on the figure's caption, and plot the complex zeros u_1 and u_2, obtained as solutions to the period relations (6.7), as a function of the modulus t_1. The existence of this numerical solution means that global regular solutions exist for h_1 and h_2 for the above assigned values. In addition, we have numerically shown that regular solutions exist for less symmetrical assignments of the real zeros, and modulus $t_2 \neq t_0$. It is an interesting problem to describe the region of parameters $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ and moduli, but this issue lies beyond the scope of this paper.

Note that the complex zero u_2 tends to the real axis as $t_1 \to 1$. This limiting case corresponds to a degeneration of the genus 2 surface, a phenomenon that will be analyzed in generality in the subsequent section.

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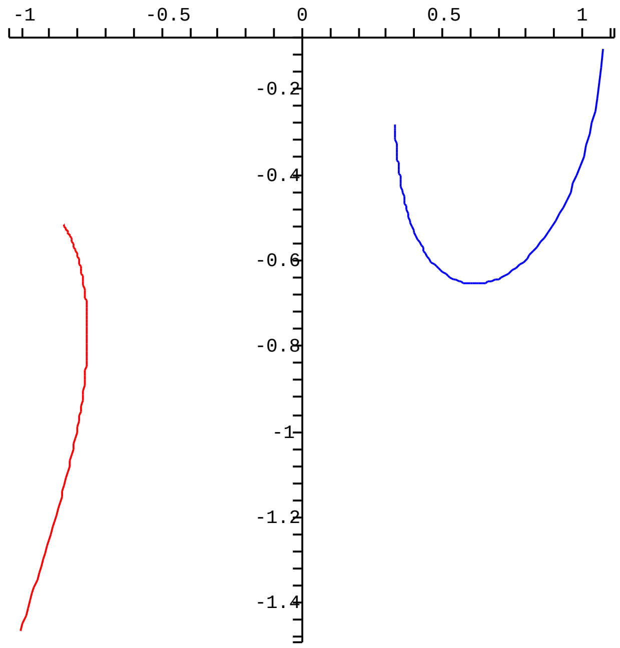

Figure 7: Genus 2 solutions for complex zeros u_1 (red) and u_2 (blue) in the lower half u-plane, for modulus t_1 in the interval $1.03 < t_1 < 2$, fixed modulus $t_2 = t_0$, and $\alpha_1 = (e_1 + e_2)/2$, $\alpha_2 = (e_3 + e_4)/2$, $\alpha_3 = 2e_5$, $\beta_1 = 2e_1$, $\beta_2 = (e_2 + e_3)/2$, and $\beta_3 = (e_4 + e_5)/2$.

6.3 Parametrization via ϑ-functions

The Thomae formulas [14] allow us to express the branch points in terms of genus two ϑ-functions. We begin by recalling some key facts about genus two ϑ-functions [14, 15]. A general spin structure is a half-integer characteristic, consisting of an array of two 2-component vectors,

$$\kappa = (\kappa' | \kappa'') \qquad \kappa' = \begin{pmatrix} \kappa'_1 \\ \kappa'_2 \end{pmatrix} \qquad \kappa'' = \begin{pmatrix} \kappa''_1 \\ \kappa''_2 \end{pmatrix} \tag{6.11}$$

and the entries $\kappa'_{1,2}$ and $\kappa''_{1,2}$ take on values 0 or 1/2 mod 1. The ϑ-function is defined in terms of the period matrix Ω, and a general complex 2-component vector ζ by

$$\vartheta[\kappa](\zeta, \Omega) = \sum_{n \in \mathbf{Z}^2} \exp\left\{\pi i (n + \kappa')^t \Omega (n + \kappa') + 2\pi i (n + \kappa')^t (\zeta + \kappa'')\right\} \tag{6.12}$$

For any spin structure κ, the combination $4(\kappa')^t \kappa''$ is an integer: even (resp. odd) spin structures correspond to $4(\kappa')^t \kappa''$ even (resp. odd).

For genus 2, there exists a sharper version of the Thomae formulas, which was heavily used in [15]. The key is a one-to-one map between the six branch points e_i, $i = 1, \cdots, 6$

with $e_6 = \infty$, and the six odd spin structures ν_i at genus 2. The relation with ϑ-functions is then simply given by

$$\frac{(e_i - e_j)(e_k - e_l)}{(e_i - e_k)(e_j - e_l)} = \frac{\mathcal{M}_{\nu_i\nu_j}\mathcal{M}_{\nu_k\nu_l}}{\mathcal{M}_{\nu_i\nu_k}\mathcal{M}_{\nu_j\nu_l}} \tag{6.13}$$

where $\mathcal{M}_{\nu_i\nu_j}$ is a modular form of weight 2, defined by

$$\mathcal{M}_{\nu_i\nu_j} = \partial_1\vartheta[\nu_i](0,\Omega)\partial_2\vartheta[\nu_j](0,\Omega) - \partial_2\vartheta[\nu_i](0,\Omega)\partial_1\vartheta[\nu_j](0,\Omega) \tag{6.14}$$

An alternative formula for $\mathcal{M}_{\nu_i\nu_j}$, with $i \neq j$, is in terms of even spin structures, but is given only up to an overall sign (familiar from the standard Thomae formulas of [14]),

$$\mathcal{M}^2_{\nu_i\nu_j} = \pi^4 \prod_{k\neq i,j} \vartheta[\nu_i + \nu_j + \nu_k](0,\Omega)^2 \tag{6.15}$$

Upon choosing $e_4 = 0$, $e_5 = 1$, and $e_6 = \infty$, by SL$(2,\mathbf{R})$ invariance, and $i = 4$, $k = 5$ and $l = 6$, the cross ratio formula yields an explicit formula for the 3 remaining real moduli,

$$e_j = \frac{\mathcal{M}_{\nu_j\nu_4}\mathcal{M}_{\nu_5\nu_6}}{\mathcal{M}_{\nu_j\nu_6}\mathcal{M}_{\nu_5\nu_4}} \tag{6.16}$$

Note that this formula actually holds for all possible values of $j = 1, \cdots, 6$.

The period integrals that enter into the period relations (6.7) are half $\mathcal{A}$- and $\mathcal{B}$-cycle integrals of Abelian differentials. We shall declare the following correspondence,

$$\begin{aligned} &[e_2, e_1] \sim \mathcal{A}_1 \qquad\qquad && [e_4, e_3] \sim \mathcal{A}_2 \\ &[e_3, e_2] \sim \mathcal{B}_1 && [e_5, e_4] \sim \mathcal{B}_2 \end{aligned} \tag{6.17}$$

We define the following periods for $k = 0, 1, 2, 3$,

$$\begin{aligned} &\int_{e_2}^{e_1} \frac{u^k du}{s(u)} = K_{1,k} \qquad\qquad && \int_{e_4}^{e_3} \frac{u^k du}{s(u)} = K_{2,k} \\ &\int_{e_3}^{e_2} \frac{u^k du}{s(u)} = L_{1,k} && \int_{e_5}^{e_4} \frac{u^k du}{s(u)} = L_{2,k} \end{aligned} \tag{6.18}$$

The periods L for $k = 0, 1$ are given in terms of the period matrix Ω and the periods K by the relations,

$$\Omega = \frac{1}{K_{1,1}K_{2,0} - K_{1,0}K_{2,1}} \begin{pmatrix} K_{2,0}L_{1,1} - K_{2,1}L_{1,0} & K_{2,0}L_{2,1} - K_{2,1}L_{2,0} \\ K_{1,1}L_{1,0} - K_{1,0}L_{1,1} & K_{1,1}L_{2,0} - K_{1,0}L_{2,1} \end{pmatrix} \tag{6.19}$$

The K periods cancel out of the set of four period relations (6.7), just as an overall factor of ω_1 cancelled out of the genus 1 period relations.

The periods K and L for $k = 2, 3$ are analogous to the quantities $\zeta(\omega_1)$ and $\zeta(\omega_3)$ encountered at genus 1. They are the periods of Abelian differentials with poles of order 2 and 4 at the branch point $e_6 = \infty$. We are aware of no known explicit formulas for these periods in terms of ϑ-functions, but suspect that such formulas could be derived with the help of the conversion formulas between the ϑ-function and hyperelliptic representations obtained in [15].

The differentials ∂h_1, ∂h_2 as well harmonic functions may also be expressed in terms ϑ-functions. The key ingredient is the identification of the Abelian differentials with double poles at the branch points with their form in terms of ϑ-functions, a result available through the use of the prime form [16]. Since the definition of the prime form involves the Abelian integrals of the first kind anyway, it is unclear how much would be gained from this alternative expression, and we shall therefore suppress these formulas.

6.4 Continuity argument for existence of solutions at all genera

The solutions for different genera form a connected set and the lower genus solutions may be obtained as a limit of higher genus solutions, by collapsing branch cuts. For example, a genus g solutions, with branch points $e_1, \cdots, e_{2g+1}$ and $e_{2g+2} = \infty$, real zeros $\alpha_1, \cdots \alpha_{g+1}$, and $\beta_1 \cdots, \beta_{g+1}$, and complex zeros $u_1, \cdots, u_g$, is smoothly connected to the genus $g-1$ solution by letting

$$\begin{aligned} e_{2b},\, e_{2b-1} &\to x \\ \alpha_b, \beta_b, u_b, \bar{u}_b &\to x \end{aligned} \tag{6.20}$$

for some $1 \leq b \leq g$. In this limit, the resulting differentials ∂h_1 and ∂h_2 become independent of the point x, and collapse onto the genus $g-1$ solution, obtained from the above genus g solution with the branch points e_{2b}, e_{2b-1} and the zeros $\alpha_b, \beta_b, u_b, \bar{u}_b$ removed. This limit is completely smooth. Importantly, the limit is a *local* process, which is largely insensitive to the global properties of Σ.

Now consider the reverse problem. If a genus $g-1$ solution exists, the question is naturally raised as to whether a genus g solution exists *in an open neighborhood of parameter space of the genus $g-1$ solution.* In other words, can an extra genus be "turned on" by inserting 2 extra branch points e_{2g+1}, e_{2g+2} and adding zeros $\alpha_g, \beta_g, u_g, \bar{u}_g$? We have investigated this question numerically for genus 1 and 2, and have found an affirmative answer to this question. We believe that it should be possible to prove analytically that a genus g solution will indeed exist in a sufficiently small open neighborhood of any genus $g-1$ solution.

One key ingredient in the above conjecture is the necessary condition that, as a branch cut collapses to a point x, exactly one complex zero converges to the real axis and collapses to x as well. This collapse phenomenon is studied in the next section, where it is shown to hold in general. It is also shown there, that, if one allows for mild singularities in the 10-dimensional geometry, such as those produced by D5 and NS5 branes in the probe limit, then only one of the real zeros, either α or β is required to collapse to x as well, but not both. Thus, allowing for such probe limit singularities, a higher-dimensional parameter space of degenerations is allowed.

7. Collapse of branch cuts and D- and NS-branes

A direct study of the collapse of branch cuts from the explicit hyper-elliptic solution is complicated by the fact that the splitting off of pole terms in h_1 and h_2, between the functions $p_{1,2}$ and $q_{1,2}$ as in formula (4.31) introduces artificial divergences in $p_{1,2}$ and $q_{1,2}$ which however cancel in the functions h_1 and h_2. For this reason, we shall take first the limit of collapsing branch cuts on the differentials ∂h_1 and ∂h_2, where this limit exists consistently, and then re-examine the questions of regularity.

7.1 The case of genus 1

For genus 1, the branch points and zeros of Q_1 and Q_2 are subject to the following ordering, $\alpha_2 < e_3 < \beta_2 < e_2 < \alpha_1 < e_1 < \beta_1$. We shall take the limit where $e_1 - e_2 \to 0$, which clearly forces also $e_1 - \alpha_1 \to 0$. Without loss of generality, we make an overall translation by e_3, and set $e_3 = 0$, which is the position of the only remaining branch point, and we use the designation $e_1, e_2, \alpha_1 \to k^2 > 0$, with $k > 0$, and $\alpha = \alpha_2$. Thus, we have the ordering

$$\alpha < 0 < \beta_2 < k^2 < \beta_1 \tag{7.1}$$

and the differentials are

$$\begin{aligned} \partial h_1 &= -i\,\frac{(u-u_1)(u-\bar u_1)(u-\alpha)}{(u-k^2)^2}\,\frac{du}{\sqrt{u}^3} \\ \partial h_2 &= -\,\frac{(u-u_1)(u-\bar u_1)(u-\beta_1)(u-\beta_2)}{(u-k^2)^3}\,\frac{du}{\sqrt{u}^3} \end{aligned} \tag{7.2}$$

As a smooth limit of the elliptic case with the ordering of $\alpha, k^2, \beta_1, \beta_2$ prescribed above, the regularity condition $W \leq 0$ is automatic. To uniformize the square root, we introduce a new coordinate $u = w^2$, which, in view of $\mathrm{Im}(u) < 0$, takes values in the second quadrant,

$$\Sigma \equiv \{w;\ \mathrm{Re}(w) < 0,\ \mathrm{Im}(w) > 0\} \tag{7.3}$$

To work out the vanishing Dirichlet conditions on h_1 and h_2, and insist on their positivity, we decompose the fractions into elementary poles, and we have,

$$\begin{aligned} \partial h_1 &= -2i\left[1 + \frac{A_1}{w^2} + \frac{2kB_1}{w^2-k^2} + \frac{2C_1(w^2+k^2)}{(w^2-k^2)^2}\right] dw \\ \partial h_2 &= -2\left[1 + \frac{A_2}{w^2} + \frac{2kB_2}{w^2-k^2} + \frac{2C_2(w^2+k^2)}{(w^2-k^2)^2} + \frac{2D_2(3w^2k+k^3)}{(w^2-k^2)^3}\right] dw \end{aligned} \tag{7.4}$$

It is straightforward to integrate these (up to an additive constant, which must vanish by the vanishing Dirichlet conditions),

$$\begin{aligned} h_1 &= -2i(w-\bar w)\left[1 + \frac{A_1}{|w|^2} + \frac{2C_1(|w|^2+k^2)}{|w^2-k^2|^2}\right] - 2iB_1 \ln\frac{(w-k)(\bar w+k)}{(w+k)(\bar w-k)} \\ h_2 &= -2(w+\bar w)\left[1 - \frac{A_2}{|w|^2} + 4C_2\frac{|w|^2-k^2}{|w^2-k^2|^2} + 4kD_2\frac{|w|^2(w^2+\bar w^2 - w\bar w - 2k^2)+k^4}{|w^2-k^2|^3}\right] \\ &\qquad -2B_2 \ln\frac{|w-k|^2}{|w+k|^2} \end{aligned} \tag{7.5}$$

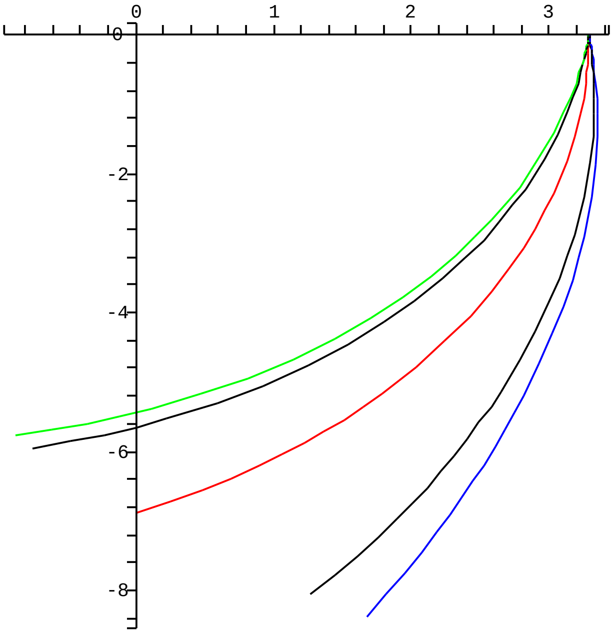

Figure 8: Genus 1 collapsing branch cut, starting with a square torus at the top of each curve, to $e_1 - e_2 \to 0$ at the real axis, for $\alpha_1 = (e_1 + e_2)/2$, $\alpha_2 = 2e_3$, $\beta_1 = 2e_1$ and various values of $\beta_2 = e_3 + x(e_2 - e_3)$, given by $x = 0.99$ (blue), $x = 0.90$ (top black), $x = 0.50$ (red), $x = 0.10$ (bottom black), and $x = 0.01$ (green).

For $w \in \Sigma$, we have $-2i(w - \bar{w}) > 0$ and $-2(w + \bar{w}) > 0$. Therefore, positivity of h_1 and h_2 at the poles $w = 0, -k$ (note that $w = +k \notin \Sigma$) requires that

$$\begin{aligned} &A_1 > 0 \qquad B_1 = 0 \qquad C_1 \geq 0 \\ &A_2 < 0 \qquad kB_2 \leq 0 \qquad C_2 = D_2 = 0 \end{aligned} \tag{7.6}$$

The requirements on A_1, A_2, C_1 are obvious. The requirement $B_1 = 0$ stems from the fact that as w crosses the point $w = -k$, the logarithms pick up an additive contribution of $2\pi i$, so that the left and the right Dirichlet boundary values of h_1 differ by $2\pi B_1$. But regularity of the solution requires that all Dirichlet boundary values vanish, so that we must have $B_1 = 0$. Finally, it is easy to see that the sign of the numerator of the D_2 term depends on how the point $w = -k$ is being approached. Setting $\mathrm{Re}(w) = -k$ and varying its imaginary part produces a negative numerator, while setting $\mathrm{Im}(w) = 0$ and varying its real part produces a positive numerator. Since this is the leading singularity at $w = -k$, we must have $D_2 = 0$. Given $D_2 = 0$, the term in C_2 remains as the leading singularity and the same argument applies to conclude that $C_2 = 0$, which concludes the proof of (7.6).

Obtaining the values of $A_1, B_1, C_1, A_2, B_2, C_2, D_2$ from those of $k, \alpha, \beta_1, \beta_2$, we have

$$
\begin{aligned}
A_1 &= -\alpha \frac{|u_1|^2}{k^4} & A_1 + 2C_1 &= -u_1 - \bar{u}_1 - \alpha + 2k^2 \\
A_2 &= -\frac{\beta_1\beta_2|u_1|^2}{k^6} & A_2 + 2kB_2 &= -u_1 - \bar{u}_1 - \beta_1 - \beta_2 + 3k^2 \\
C_1 &= \frac{|k^2 - u_1|^2(k^2-\alpha)}{4k^4} & B_1 &= 0
\end{aligned} \tag{7.7}
$$

together with the requirement that $(u-u_1)(u-\bar{u}_1)(u-\beta_1)(u-\beta_2)$ have a double zero in u at $u = k^2$, which is equivalent to $C_2 = D_2 = 0$. Note that the conditions $A_1 > 0, A_2 < 0, C_1 > 0$ of (7.6) follow directly from the above identifications and the ordering (7.1).

To solve the remaining conditions, it is helpful to first solve the $C_2 = D_2 = 0$ condition. There are two cases. Either the double zero of $(u-u_1)(u-\bar{u}_1)(u-\beta_1)(u-\beta_2)$ is simply $u_1 = k^2$, or it is not: $u_1 \neq k^2$.

7.1.1 The case $u_1 \neq k^2$

In this case, the double zero requires $\beta_1 = \beta_2 = k^2$, and we have $2k^3 B_2 = |k^2 - u_1|^2 > 0$, and hence the condition $kB_2 \leq 0$ can never be satisfied. Thus, this case is simply ruled out.

7.1.2 The case $u_1 = k^2$

This leads to drastic simplifications, and we have

$$
\begin{aligned}
A_1 &= -\alpha & B_1 &= C_1 = C_2 = D_2 = 0 \\
A_2 &= -\frac{\beta_1\beta_2}{k^2} & B_2 &= \frac{1}{2k^3}(k^2-\beta_1)(k^2-\beta_2)
\end{aligned} \tag{7.8}
$$

We see that the condition $kB_2 \leq 0$ is now automatic as well. The solution may be written down explicitly. The harmonic functions are given by

$$
\begin{aligned}
h_1 &= -2i(w-\bar{w})\left[1 - \frac{\alpha}{|w|^2}\right] \\
h_2 &= -2(w+\bar{w})\left[1 + \frac{\beta_1\beta_2}{k^2|w|^2}\right] + \frac{(\beta_1 - k^2)(k^2-\beta_2)}{k^3}\ln\frac{|w-k|^2}{|w+k|^2}
\end{aligned} \tag{7.9}
$$

Since $w \in \Sigma$ of (7.3), and using the ordering (7.1), it is immediate that $h_1, h_2 > 0$ everywhere in $w \in \Sigma$. In particular, the ln is always positive since for $w \in \Sigma$, we have $|w-k|^2 > |w+k|^2$. Notice that, if we also have $\beta_1 = k^2$ or $\beta_2 = k^2$, then we recover the Janus solution.

7.2 General collapse of a branch cut

Having derived the behavior under collapse of a branch cut in the genus 1 case, we are now in a position to generalize the result to the collapse of a branch cut for general genus. The reason that such a result can be obtained is that the regularity and positivity conditions are essentially local conditions, largely insensitive to the global properties of the surface Σ.

Below, we shall show the following general result. *The collapse of any branch cut is accompanied by the convergence of precisely one of the complex zeros u_a to the real axis, and more specifically, to the location of the collapsing branch cut.*

For simplicity, we shall assume that two consecutive branch points e_{2b} and e_{2b-1} collapse to a zero of type α, and that the adjacent branch points e_{2b+1} and e_{2b-2} remain a finite distance away from α. (If adjacent branch points are allowed to collapse onto α as well, we are dealing with a multiple degeneration of the Riemann surface. The same methods, to be explained below, can also be adapted to cover those cases.)

The corresponding branch cut $[e_{2b}, e_{2b-1}]$ contains a single zero α_b of Q_1. (The argument may readily be carried over to an interval $[e_{2b+1}, e_{2b}]$ which would contain a single zero β_{b+1} of Q_2 instead.) Since $\alpha_b \in [e_{2b}, e_{2b-1}]$, this means that also $\alpha_b \to \alpha$. The resulting ordering of the collapsed interval is thus,

$$\alpha_{g+1} < e_{2g+1} < \cdots < e_{2b+1} < \beta_{b+1} < \alpha < \beta_b < e_{2b-2} < \cdots < e_1 < \beta_1 \tag{7.10}$$

Having taken the limit $e_{2b}, e_{2b-1}, \alpha_b \to \alpha$, we shall now study the problem locally around $u \sim \alpha$. To do so, it is convenient to define the following quantities,

$$\begin{aligned} s(u)^2 &= (u-\alpha)^2 \tilde{s}(u)^2 \\ Q_1(u) &= (u-\alpha)\tilde{Q}_1(u) \end{aligned} \tag{7.11}$$

where

$$\begin{aligned} \tilde{s}(u)^2 &= (u - e_{2g+1}) \prod_{j=1,\, j\neq b}^{g} (u-e_{2j-1})(u-e_{2j}) \\ \tilde{Q}_1(u) &= \prod_{a=1,\, a\neq b}^{g} (u-\alpha_a) \end{aligned} \tag{7.12}$$

and recast the differentials ∂h_1 and ∂h_2 in terms of the following functions,

$$\begin{aligned} \partial h_1 &= -iM_1(u)\frac{du}{(u-\alpha)^2} \qquad & M_1(u) &\equiv \frac{P(u)\tilde{Q}_1(u)}{\tilde{s}(u)^3} \\ \partial h_2 &= -M_2(u)\frac{du}{(u-\alpha)^3} \qquad & M_2(u) &\equiv \frac{P(u)Q_2(u)}{\tilde{s}(u)^3} \end{aligned} \tag{7.13}$$

Note that, because of the extra zero in Q_1 at $u = \alpha$, the differential ∂h_1 has a pole of order 2 while ∂h_2 has a pole of order 3. Also, it is readily verified that, since $e_{2b+1} < \alpha < e_{2b-2}$, we have $\tilde{s}(\alpha)^2 > 0$, so that $\tilde{s}(u)$, and thus $M_1(u)$ and $M_2(u)$ are real for u real and in the neighborhood of α. Expanding the differentials in powers of w, with $u = \alpha + w$, and integrating to obtain the harmonic functions h_1 and h_2, we find,

$$\begin{aligned} h_1 &= M_1(\alpha)\frac{2\mathrm{Im}(w)}{|w|^2} - iM_1'(\alpha)\ln\frac{w}{\bar{w}} + \text{regular} \\ h_2 &= M_2(\alpha)\frac{\mathrm{Re}(w)^2 - \mathrm{Im}(w)^2}{|w|^4} + M_2'(\alpha)\frac{2\mathrm{Re}(w)}{|w|^2} - M_2''(\alpha)\ln|w| + \text{regular} \end{aligned} \tag{7.14}$$

Positivity of h_2 clearly requires that

$$M_2(\alpha) = M_2'(\alpha) = 0 \qquad\qquad M_2''(\alpha) \geq 0 \tag{7.15}$$

Positivity at the leading singularity of h_1 requires that (recall that u and thus w are in the lower half-plane, with $\mathrm{Im}(w) < 0$),

$$M_1(\alpha) \leq 0 \tag{7.16}$$

There is also a constraint on the $M_1'(\alpha)$ term which is more subtle. The function h_1 must obey vanishing Dirichlet boundary conditions on both sides of α. The term proportional to $M_1'(\alpha)$ introduces a discontinuity in h_1 along the boundary, adding to h_1 the quantity $2\pi M_1'(\alpha)$ as w moves across α along the real axis. No such discontinuity is allowed with vanishing Dirichlet boundary conditions, and no such discontinuity can be compensated for by the regular terms, which have not been explicitly exhibited. Therefore, we have the final condition that

$$M_1'(\alpha) = 0 \tag{7.17}$$

Clearly, condition (7.15) requires that $M_2(u)$ must have a double zero at $u = \alpha$. There are only two different ways of achieving a double zero. (It is here that we use the extra assumption, stated above, that the adjacent branch points e_{2b+1} and e_{2b-2} remain a finite distance away from α.)

7.2.1 The case $P(\alpha) \neq 0$

In this case, we must have $Q_2(\alpha) = 0$, which requires that $\beta_b = \beta_{b+1} = \alpha$. We shall now calculate the sign of $M_2''(\alpha) = P(\alpha)Q_2''(\alpha)/\tilde{s}(\alpha)^3$. Analyzing the position of α relative to the branch points and zeros of Q_2'', we readily find,

$$\begin{aligned} \mathrm{sign}(\tilde{s}(\alpha)) &= (-1)^b \\ \mathrm{sign}(Q_2''(\alpha)) &= (-1)^{b-1} \end{aligned} \tag{7.18}$$

It follows that $M_2''(\alpha) < 0$; the inequality is strict because we have $P(\alpha) > 0$, as well as $Q_2''(\alpha), \tilde{s}(\alpha) \neq 0$. But this result is contradictory to the third result in (7.15). Therefore, the case $P(\alpha) \neq 0$ is ruled out, and we must have instead,

7.2.2 The case $P(\alpha) = 0$

Since the zeros of $P(u)$ come in complex conjugate pairs, a real zero α must be a double zero of P. It then follows that the first and second conditions of (7.15), as well as (7.16) and (7.17) hold automatically. Since we now have $\mathrm{sign}(Q_2(\alpha)) = (-1)^b$ and $P''(\alpha) > 0$, it is easy to see that the third condition in (7.15) is also satisfied.

7.3 Complete collapse of the higher genus case

When branch cuts are being collapsed at higher genus, there are a priori two options for the remaining branch point: it can have even or odd index. If the index is even, e_{2r}, for

$1 \leq r \leq g$, then either the branch point e_1 or the branch point e_{2g+1} should be collapsed onto the branch point at ∞ in order to obtain maximal collapse. If the index is odd, e_{2r-1}, $1 \leq r \leq g+1$, then the point ∞ remains a branch point. It is the latter case we shall discuss in detail here, the other case is analogous. As before, we shift and set $e_{2r-1} = 0$. It is clear that the branch cuts to the left of 0 collapse with a β-zero, while the ones to the right collapse with an α-zero. Following the arguments of subsection 7.2, each of the g complex zeros must collapse to one of the g collapsing branch cuts. We shall denote the negative collapsed branch cuts $-l_j^2$ and the positive collapsed branch cuts by $+k_i^2$, with $i = 1, \cdots, m = r-1$ and $j = 1, \cdots, n = g-r+1$. In view of the above considerations, the form of the differentials reduces to

$$\begin{aligned} \partial h_1 &= -2idw \left[1 + \frac{C_0}{w^2} + \sum_{j=1}^{n} \frac{C_j}{w^2 + l_j^2} \right] \\ \partial h_2 &= -2dw \left[1 + \frac{D_0}{w^2} + \sum_{i=1}^{m} \frac{D_i}{w^2 - k_i^2} \right] \end{aligned} \tag{7.19}$$

for $k_i, l_j > 0$. It is straightforward to compute W, and check that the condition $W < 0$ is fulfilled as soon as we have $C_0 - D_0 > 0$, and

$$\begin{aligned} C_j > 0 \quad j = 1, \cdots, m \\ D_i < 0 \quad i = 1, \cdots, n \end{aligned} \tag{7.20}$$

The functions are also readily evaluated,

$$\begin{aligned} h_1 &= -2i(w - \bar{w}) \left[1 + \frac{C_0}{|w|^2} \right] + \sum_{j=1}^{n} \frac{C_j}{l_j} \ln \frac{|w + il_j|^2}{|w - il_j|^2} \\ h_2 &= -2(w + \bar{w}) \left[1 - \frac{D_0}{|w|^2} \right] - \sum_{i=1}^{m} \frac{D_i}{k_i} \ln \frac{|w - k_i|^2}{|w + k_i|^2} \end{aligned} \tag{7.21}$$

It is immediate that these functions satisfy vanishing Dirichlet boundary conditions, $h_1 = 0$ whenever w is real, and $h_2 = 0$ whenever w is purely imaginary. The conditions $h_1, h_2 > 0$ require the stronger inequalities $C_0 > 0$ and $D_0 < 0$. For $w \in \Sigma$, using the inequalities, $|w + il_j|^2 > |w - il_j|^2$ and $|w - k_i|^2 > |w + k_i|^2$, and the fact that $C_j > 0$ while $D_i < 0$, as already assumed in (7.20), it is immediate that $h_1, h_2 > 0$. This completes our proof of regularity of the solutions with poles on the boundary of the Riemann surface Σ.

7.4 The probe limit

In this subsection, the behavior of the solutions with collapsed branch points will be analyzed near the pole $w = -k$. Note that the two branch points collapse at a zero α_b of ∂h_1. The goal is to identify the singular solution in this limit with the metric of the probe NS5-brane. The limit when two branch points collapse near a zero β_b of ∂h_2 corresponds to a probe D5-brane.

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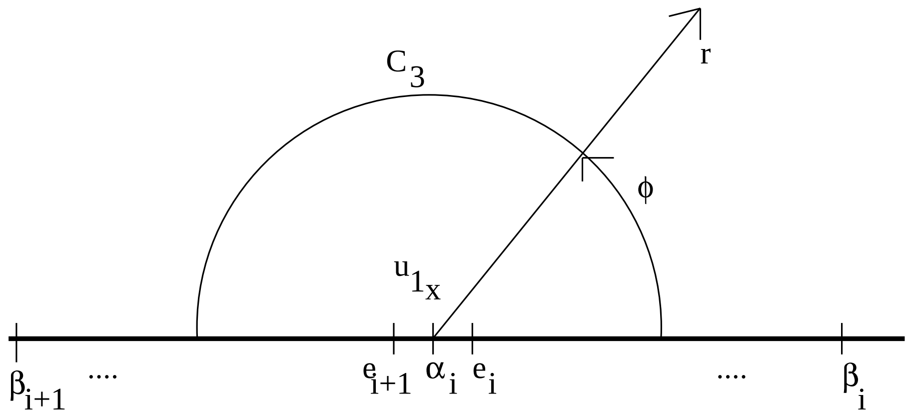

Figure 9: Local coordinates near two collapsed branch points.

We define $w = -k + z$ and introduce a new coordinate $z = re^{i\psi}$. The expansion near the collapsed branch points is around $r = 0$. The harmonic functions (7.9) have the expansion

$$h_1 = 4c_1 r \sin\psi \qquad h_2 = 2d_0 - 2d_1 \ln r^2 \tag{7.22}$$

where $d_0, d_1, c_1 > 0$ and subleading terms in r and $\ln r$ are dropped. In this limit the solution becomes

$$\begin{aligned} e^{4\phi} &= \frac{d_1^2}{c_1^2}\frac{|\ln(r)|}{r^2} + \cdots \\ \rho^2 &= 4\sqrt{c_1 d_1}\,\frac{1}{r^{\frac{3}{2}}|\ln(r)|^{\frac{1}{4}}} + \cdots \\ f_1^2 &= 4\sqrt{c_1 d_1}\,(\sin\psi)^2\frac{r^{\frac{1}{2}}}{|\ln(r)|^{\frac{1}{4}}} + \cdots \\ f_2^2 &= 4\sqrt{c_1 d_1}\,r^{\frac{1}{2}}|\ln(r)|^{\frac{1}{4}} + \cdots \\ f_4^2 &= 4\sqrt{c_1 d_1}\,r^{\frac{1}{2}}|\ln(r)|^{\frac{1}{4}} + \cdots \end{aligned} \tag{7.23}$$

Note that the range of the angular parameter is $\psi \in [0, \pi]$. The behavior of the metric factors f_1 and f_2 given in (7.23) implies that for fixed r the compact part of the metric is $S^3 \times S^2$. The leading terms for the 2-form potential $B_{(2)}$ are given by

$$\begin{aligned} b_1 &= -8d_1\psi + 2\sin 2\psi + o\left(\frac{1}{|\ln(r)|}\right) \\ b_2 &= 16c_1 r\cos\psi + o\left(\frac{r}{|\ln(r)|}\right) \end{aligned} \tag{7.24}$$

The function b_1 is associated with the NSNS two form potential and remains finite as $r \to 0$. This implies that there is a non-zero NSNS flux through the three sphere parameterized by ψ and S_1^2. The flux is given by

$$\lim_{r\to 0}\int_{S^3} H_3 = \lim_{r\to 0}\int_{S^3} db_1\wedge \hat e^{45}$$
$$= \lim_{r\to 0} 4\pi\left(b_1\Big|_{\psi=\pi} - b_1\Big|_{\psi=0}\right)$$
$$= -32\pi^2 d_1 \tag{7.25}$$

Clearly, the limit of collapsing branch points is singular. There is, however, a natural interpretation of this singular behavior. The fact that the dilaton goes to infinity, that there is a non-vanishing NSNS 3-form flux through an S^3 and that metric factors for the S_2^2 and AdS_4 behave in the same way all indicate that the singularity can be attributed to the presence of a NS5-brane source with worldvolume $AdS_4\times S_2^2$. At this point, only the flat space NS-brane metric is known. A construction of a solution of the supergravity equations in this background with an NS5-brane source and a detailed comparison of the solutions is beyond the scope of this paper.

A similar analysis can be carried out for the collapse of two branch points near a zero of ∂h_2. We will not give the details of this limit here since they are analogous to the ones presented above. The dilaton goes to zero, the coordinate ψ and S_2^2 produce a non-trivial S^3 and there is non-vanishing RR 3-form flux through this three sphere. Hence the interpretation of this singular solution is given by a probe D5-brane with worldvolume $S_1^2\times AdS_4$.

8. The AdS/CFT dual interface gauge theory

The holographic interpretation [17–19] (for reviews, see [20, 21]) of the original Janus solution as well as the $\mathcal{N}=1$ and $\mathcal{N}=4$ supersymmetric generalizations are given by interface conformal field theories respectively in [22, 23], and [1]. The dual four-dimensional field theory lives on two four-dimensional half-spaces glued together at a three-dimensional interface.

The boundary structure is found by determining where the various components of the metric blow up. Using the Poincaré patch metric for the AdS_4 metric the ten-dimensional metric is given by

$$ds^2 = 4\rho^2|dw|^2 + f_4^2\left(\frac{-dt^2+dx_1^2+dx_2^2+dz^2}{z^2}\right) + f_1^2 ds_{S_1^2} + f_2^2 ds_{S_2^2} \tag{8.1}$$

It was shown in section 4 that the metric is completely regular for appropriate choices of the parameters of the genus g hyperelliptic Ansatz. The only place where a metric factor blows up is at the branch points e_i, for $i=1,2,\cdots,2g-1$ (and infinity), where f_4 diverges. There is an additional divergence of the AdS_4 metric when $z\to 0$. The boundary structure becomes clearer if one rewrites the metric as follows

$$ds^2 = \frac{f_4^2}{z^2}\left\{-dt^2+dx_1^2+dx_2^2+dz^2+\frac{z^2}{f_4}\Big(4\rho^2|du|^2+f_1^2 ds_{S_1^2}+f_2^2 ds_{S_2^2}\Big)\right\} \tag{8.2}$$

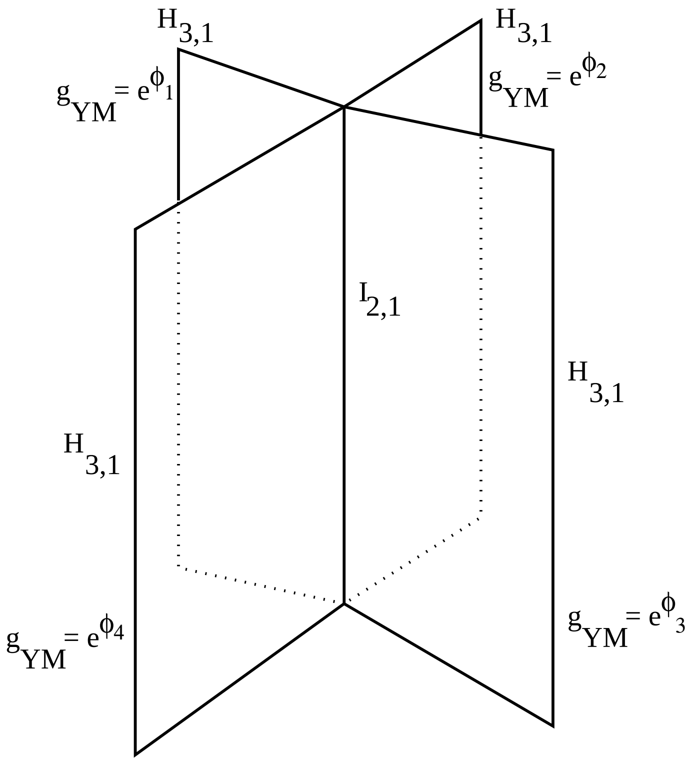

Figure 10: Dual interface theory for the genus 1 multi-Janus solution

Near the $2g+2$ branch points, f_4 diverges and there are $2g+2$ different $3+1$-dimensional half-spaces $H_{1,3}^{(i)}$, $i = 1, 2, \cdots 2g+2$ each spanned by t, x_1, x_2, z. Each one of these spaces has a boundary at $z = 0$. Since at $z = 0$ the overall metric factor blows up, too, there is an additional boundary component given by a $2+1$-dimensional space $I_{1,2}$ spanned by t, x_1, x_2.

The behavior of the fields near the boundary was worked out in subsection 4.11. The standard holographic relations [19] allow the identification of the dual conformal field theory. The detailed analysis is the same as the one presented in [1] and will not be repeated here. The result that the holographic theory is an interface theory defined by $2g+2$ copies of $\mathcal{N} = 4$ SYM living on (independent) $3+1$-dimensional half-spaces $H_{1,3}^{(i)}$. The boundaries of the half-spaces are all connected via a $2+1$-dimensional interface $I_{1,2}$.

The SYM theories on the i-th half-space (which is associated with the branch point e_i), has the coupling constant which can be determined from (4.46).

$$g_{YM}^{(i)} = \left| \frac{Q_2(e_i)}{Q_1(e_i)} \right| \tag{8.3}$$

In figure 10, we sketch the situation for the $g = 1$ case which has four boundary components. Note that in figure 10, the four-dimensional spaces are embedded in a higher-dimensional space. This is mainly done for clarity, but is also natural from a holographic perspective. A different interpretation uses a "folding trick" employed in [24] where $2g+2$ copies of $\mathcal{N} = 4$

SYM fields with coupling constants (8.3) exist independently on a single $3+1$-dimensional half-space and only interact on the $2+1$-dimensional interface.

9. Discussion

In this paper, we have found an infinite family of non-singular solutions to the half-BPS equations and Type IIB supergravity field equations on $AdS_4 \times S^2 \times S^2 \times \Sigma$ with $SO(2,3) \times SO(3) \times SO(3)$ symmetry. The general local solution was constructed in the companion paper [1], and is specified by two harmonic functions h_1 and h_2 on a Riemann surface Σ with boundary. The class of non-singular solutions found in this paper is parameterized by a genus g hyper-elliptic surface Σ with boundary. The branch points of Σ are restricted to lie on the real axis. Global regularity is obtained by enforcing the conditions (R1-R7) of subsection 3.5 on the harmonic functions h_1 and h_2. The ordering (4.40) of the $2g+2$ branch points and $g+1$ branch cuts on the real line is a one-dimensional analog of the coloring of the two-dimensional plane in the "bubbling AdS" solutions[8] of [5]. The conditions (R1-R7) in addition require that the real zeros of the differentials ∂h_1 and ∂h_2 obey a relative order amongst themselves and with the branch points.

There are several directions for further research concerning the half-BPS interface theories. Our regular solutions were found amongst the general local solution of [1] by imposing regularity and topology conditions. The topology conditions we enforced guarantee that there are asymptotic regions, where the geometry approaches $AdS_5 \times S^5$. It would be interesting to investigate whether these conditions are necessary for regularity. In particular, is it possible to find solutions on Riemann surfaces with topology different from the ones we have used ? For example, can one find regular solutions with $h_1, h_2 > 0$ inside Σ, but with $W > 0$ as well ? What is their topology ?

It would also be interesting to analyze the proposed dual interface CFT of the multi-Janus solutions in more detail; in particular how does one picture the $2g+2$ half-spaces glued together over a three-dimensional interface, and what are the properties of the corresponding interface super Yang-Mills theory ? For example, one feature of our construction is that the number of half-spaces always has to be even. It is an interesting question whether this is required for the preservation of supersymmetry on the interface along the lines of the arguments of [3], or is already required in order to have a consistent Dirac equation for the fermion fields in the problem.

A new feature of the multi-Janus solutions (which has genus $g \geq 1$, while the supersymmetric Janus solution had genus $g=0$) is the presence of topologically non-contractible 3-cycles, and associated non-vanishing NSNS and RR 3-form charges. This fact, and the behavior of our solution in the limit where a branch cut collapses to a point, as was discussed in subsection 7.4, shows that the solutions found in this paper are the fully back-reacted versions corresponding to the insertion of probe D5 and NS5 branes in $AdS_5 \times S^5$ considered in [25, 4, 26]. Therefore, our half-BPS interface solutions provide a new example of "delocalization". The defect conformal field theories associated with the probe 5-branes

[8]Similar observations were made in [2].

contain extra degrees of freedom coming from the strings localized at the intersection of the three and five branes. In the fully back-reacted geometry, the localized probe branes are replaced by a full non-singular geometry with flux, and the extra degrees of freedom, supported only on the defect, have disappeared. It would be interesting to investigate whether this phenomenon is connected to the observations of [27, 28].

A closely related class of supergravity solutions has $SO(2,1)\times SO(3)\times SO(5)$ symmetry and is described by an $AdS_2\times S^2\times S^4\times\Sigma$ Ansatz. The gravitational solution is interpreted as the fully back reacted geometry dual to half-BPS Wilson loops [6−8]. A detailed analysis of this solution applying the methods of this paper can be found in a further companion paper [9].

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A. No Regular solutions with complex poles and cuts

While it would seem natural to consider solutions in which poles and branch cuts appear on the inside of the Riemann surface Σ (as opposed to the branch cuts and poles on the boundary, which were explored in the main body of this paper), we shall show in this appendix that no such regular solutions exist. We begin by working out the example with a single interior pole.

A.1 Janus plus one complex pole

We work in the lower half-plane with the variable u such that $\mathrm{Im}(u) < 0$. Including a complex pole[9] p^2, with $\mathrm{Im}(p^2) < 0$ requires that we also include the complex pole $\bar{p}^2$. To keep the behavior at $u = \infty$ unchanged, we need to also include a complex zero c and its complex conjugate $\bar{c}$. Combining all these ingredients, we have the following complex pole generalization of the Janus solution,

$$
\begin{aligned}
\partial h_1 &= -i\frac{(u-\alpha)(u-c)(u-\bar{c})}{(u-p^2)(u-\bar{p}^2)\sqrt{u}^3}du \\
\partial h_2 &= -\frac{(u-\beta)(u-c)(u-\bar{c})}{(u-p^2)(u-\bar{p}^2)\sqrt{u}^3}du
\end{aligned}
\tag{A.1}
$$

Clearly, when $c = p^2$, the Janus solution is obtained as a limiting case. Negativity of W on the lower half u-plane, and positivity at the poles $u = 0, \infty$ requires that $\beta > 0$ and $\alpha < 0$, as in the Janus case. Next, we impose vanishing Dirichlet boundary conditions and positivity of h_1 and h_2 throughout the lower half-plane, for which we need the functions h_1 and h_2.

[9] The square p^2 is used here for later convenience; without loss of generality, we choose $\mathrm{Re}(p) < 0 < \mathrm{Im}(p)$.

To uniformize the square root, we change variables to $w^2 = u$ with $\mathrm{Re}(w) < 0$, and $\mathrm{Im}(w) > 0$, as in (7.3), so that $du/\sqrt{u}^3 = 2dw/w^2$, and decompose into elementary fractions,

$$\frac{(w^2-\alpha)(w^2-c)(w^2-\bar{c})}{w^2(w^2-p^2)(w^2-\bar{p}^2)} = 1 + \frac{A_0}{w^2} + \frac{A_1}{w^2-p^2} + \frac{\bar{A}_1}{w^2-\bar{p}^2}$$
$$\frac{(w^2-\beta)(w^2-c)(w^2-\bar{c})}{w^2(w^2-p^2)(w^2-\bar{p}^2)} = 1 + \frac{B_0}{w^2} + \frac{B_1}{w^2-p^2} + \frac{\bar{B}_1}{w^2-\bar{p}^2} \tag{A.2}$$

where

$$A_0 = -\alpha\frac{|c|^2}{|p|^4} > 0 \qquad A_1 = \frac{(p^2-\alpha)(p^2-c)(p^2-\bar{c})}{p^2(p^2-\bar{p}^2)}$$
$$B_0 = -\beta\frac{|c|^2}{|p|^4} < 0 \qquad B_1 = \frac{(p^2-\beta)(p^2-c)(p^2-\bar{c})}{p^2(p^2-\bar{p}^2)} \tag{A.3}$$

The functions h_1 and h_2 may then be obtained by elementary integrals. Decomposing the ratios A_1/p and B_1/p into their real and imaginary parts, we obtain,

$$\begin{aligned} h_1 &= h_1^{(0)} - 2i(w-\bar{w}) + 2iA_0\frac{\bar{w}-w}{|w|^2} + \mathrm{Re}\left(-i\frac{A_1}{p}\right)\ln\left|\frac{(w-p)(w+\bar{p})}{(w+p)(w-\bar{p})}\right|^2 \\ &\quad + i\,\mathrm{Im}\left(\frac{-iA_1}{p}\right)\ln\frac{(w-p)(\bar{w}+p)(\bar{w}+\bar{p})(w-\bar{p})}{(\bar{w}-\bar{p})(w+\bar{p})(w+p)(\bar{w}-p)} \\ h_2 &= h_2^{(0)} - 2(w+\bar{w}) + 2B_0\frac{\bar{w}+w}{|w|^2} - \mathrm{Re}\left(\frac{B_1}{p}\right)\ln\left|\frac{(w-p)(w-\bar{p})}{(w+p)(w+\bar{p})}\right|^2 \\ &\quad - i\,\mathrm{Im}\left(\frac{B_1}{p}\right)\ln\frac{(w-p)(\bar{w}-p)(\bar{w}+\bar{p})(w+\bar{p})}{(\bar{w}-\bar{p})(w-\bar{p})(w+p)(\bar{w}+p)} \end{aligned} \tag{A.4}$$

The logarithms in the last term in each expression are multiple-valued in w around the pole p. Single-valuedness of h_1 and h_2 requires that $\mathrm{Re}(A_1/p) = \mathrm{Im}(B_1/p) = 0$. These conditions are equivalent to requiring vanishing periods of both differentials ∂h_1 and ∂h_2 around the pole p. We parametrize the solutions to these equations explicitly by

$$A_1 = -ipa_1 \qquad B_1 = pb_1 \qquad a_1, b_1 \in \mathbf{R} \tag{A.5}$$

Requiring h_1 to vanish for real $w < 0$, and h_2 to vanish for imaginary w, we find that we must have $h_1^{(0)} = h_2^{(0)} = 0$. We then have the following simplified expressions,

$$\begin{aligned} h_1 &= -2i(w-\bar{w}) + 2iA_0\frac{\bar{w}-w}{|w|^2} - a_1\ln\left|\frac{(w-p)(w+\bar{p})}{(w+p)(w-\bar{p})}\right|^2 \\ h_2 &= -2(w+\bar{w}) + 2B_0\frac{\bar{w}+w}{|w|^2} - b_1\ln\left|\frac{(w-p)(w-\bar{p})}{(w+p)(w+\bar{p})}\right|^2 \end{aligned} \tag{A.6}$$

Since we have assumed that both w and p lie in the second quadrant,

$$\begin{aligned} &\mathrm{Re}(w) < 0 \quad \mathrm{Re}(p) < 0 \\ &\mathrm{Im}(w) > 0 \quad \mathrm{Im}(p) > 0 \end{aligned} \tag{A.7}$$

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we have the inequalities,

$$
\begin{aligned}
|w-p| &< |w-\bar{p}| \qquad & |w+\bar{p}| &< |w+p| \\
|w-p| &< |w+\bar{p}| \qquad & |w-\bar{p}| &< |w+p|
\end{aligned}
\tag{A.8}
$$

Thus, positivity of h_1 and h_2 will be assured for w in the second quadrant provided,

$$
a_1, b_1 \geq 0 \tag{A.9}
$$

It remains to solve (A.3) and the period relations (A.5). Combining both relations, we have,

$$
\begin{aligned}
(p^2-\alpha)(p^2-c)(p^2-\bar{c}) &= -ia_1 p^3 (p^2-\bar{p}^2) \\
(p^2-\beta)(p^2-c)(p^2-\bar{c}) &= +b_1 p^3 (p^2-\bar{p}^2)
\end{aligned}
\tag{A.10}
$$

We consider these two complex equations for a_1 and b_1 given, and view p^2 and the position of the zero c as determined by a_1, b_1, and the above period relations. Taking the ratio of the two equations, we obtain an equation for p^2,

$$
p^2 = \frac{\alpha b_1 + i\beta\alpha_1}{b_1 + ia_1} \tag{A.11}
$$

The imaginary part of p^2 is given by

$$
\mathrm{Im}(p^2) = \frac{a_1 b_1 (\beta-\alpha)}{|a_1 - ib_1|^2} \tag{A.12}
$$

Under the conditions $a_1, b_1 > 0$, and using the inequality on the real zeros $\beta - \alpha > 0$, the pole p^2 actually lies in the *upper* half-plane, contrarily to our assumptions.

Note that if we had taken p^2 in the upper half-plane instead, and p in the first quadrant, the inequalities in the first line of (A.8) would remain unchanged, but the inequalities in the last line would get reversed, with the effect of now requiring $a_1 \geq 0$ but $b_1 \leq 0$. From (A.12), we would then find that now p^2 must be in the lower half-plane, which is again in contradiction with the assumptions made. Thus, the formal solution with one complex pole can never satisfy all the necessary conditions of regularity.

A.2 The addition of general complex poles and branch cuts

The arguments presented in the preceding subsection are essentially local on the surface Σ. Thus, by analogous reasoning, the presence of multiple complex poles is also ruled out. The arguments may also be generalized to complex branch cuts.

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